

A new version of Brakke's local regularity theorem

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September 16, 2016

Abstract

Consider an integral Brakke flow (μ_t) , $t \in [0, T]$, inside some ball in Euclidean space. If μ_0 has small height, its measure does not deviate too much from that of a plane and if μ_T is non-empty, then Brakke's local regularity theorem yields that (μ_t) is actually smooth and graphical inside a smaller ball for times $t \in (C, T - C)$ for some constant C . Here we extend this result to times $t \in (C, T)$. The main idea is to prove that a Brakke flow that is initially locally graphical with small gradient will remain graphical for some time. Moreover we use the new local regularity theorem to generalise White's regularity theorem to Brakke flows.

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1 Introduction

Overview. Consider a smooth family of embeddings $F_t : M^{\mathbf{n}} \rightarrow \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ for $t \in I$, where $M^{\mathbf{n}}$ is an \mathbf{n} -dimensional manifold and I is an open interval in \mathbb{R} . The family $M_t = F_t[M^{\mathbf{n}}]$ is called a smooth mean curvature flow if

$$(1.1) \quad (\partial_t F_t(\hat{p}))^\perp = \mathbf{H}(M_t, F_t(\hat{p}))$$

for all $(t, \hat{p}) \in I \times M^{\mathbf{n}}$. Here $(\cdot)^\perp$ denotes the projection onto the normal space and \mathbf{H} denotes the mean curvature vector. This evolution equation can be generalised to \mathbf{n} -rectifiable Radon measures on $\mathbb{R}^{\mathbf{n}+\mathbf{k}}$ (see Definition 2.3). Such a weak solution will be called a Brakke flow. Here we want to show that under certain local assumptions a Brakke flow locally satisfies the smooth characterization above.

The Mean curvature flow was introduced by Brakke in his pioneering work [Bra78], where he described the evolution in the setting of geometric measure theory. This early work already contains an existence result as well as a regularity theory. His local regularity theorem states that a Brakke flow that lies in a narrow slab and consists of one sheet is locally smooth and graphical for a certain time interval. Actually the arguments in [Bra78] often contain gaps or small errors. A new rigorous proof of the regularity results was given by Kasai and Tonegawa [KT14], [Ton15]. Also the author's thesis [Lah14] offers a complete version of Brakke's regularity theory following the original approach.

A major breakthrough in the studies of mean curvature flow was the monotonicity formula found by Huisken [Hui90] for smooth flows, which later was generalised to Brakke flows by Ilmanen [Ilm95] and localised by Ecker [Eck04]. With the help of this monotonicity formula, White proved a local regularity theorem [Whi05] stating that Gaussian density ratios close to one yield curvature estimates. White's theorem is formulated for smooth mean curvature flow and can be applied to many singular situations as well, but not for arbitrary Brakke flows. Building up on White's curvature estimates, Ilmanen, Neves and Schulze [INS14] showed that a smooth mean curvature flow which is locally initially graphical with small gradient remains graphical for some time. For related gradient and curvature estimates see [EH89], [EH91], [CM03], [Wan04], [CY07], [BH12], [Lah15].

Existence results for generalized solutions of mean curvature flow can be found in [Bra78], [CGG91], [ES91], [ES94], [Ilm94], [KT15]. For an introduction to weak mean curvature flow we recommend the work of Ilmanen [Ilm94] which also points out the similarities between Brakke flow and level set flow. We also want to mention the book of Ecker [Eck04] as a good reference for smooth mean curvature flow and regularity up to the first singular time.

Results of the present article. We consider Brakke flows of \mathbf{n} -rectifiable Radon measures in $\mathbb{R}^{\mathbf{n}+\mathbf{k}}$ see Definition 2.3 for the details. All Brakke flows considered here are assumed to be integral. The constants below may depend on \mathbf{n} and \mathbf{k} .

Our main result is a new version of Brakke's local regularity theorem [Bra78, Thm. 6.10, Thm. 6.11], see also Kasai and Tonegawa [KT14, Thm. 8.7]. Our statement says that a non-vanishing Brakke flow which initially locally lies in a narrow slab and consists of less than two sheets, becomes graphical in a small neighbourhood.

1.1 Theorem. *There exists an $\alpha_0 \in (0, 1)$ and for every $\lambda \in (0, 1)$ there exists a $\gamma_0 \in (0, 1)$ such that the following holds:*

Let $\gamma \in [0, \gamma_0]$, $\rho \in (0, \infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1 + \gamma^{\alpha_0} \rho^2, t_1 + \alpha_0 \rho^2]$, $a = (\hat{a}, \tilde{a}) \in \mathbb{R}^n \times \mathbb{R}^k$ and let $(\mu_t)_{t \in [t_1, t_2]}$ be a Brakke flow in $\mathbf{B}(a, 2\rho)$ with $a \in \text{spt} \mu_{t_2}$. Suppose we have

$$(1.2) \quad \text{spt} \mu_{t_1} \cap \mathbf{B}(a, 2\rho) \subset \{(\hat{x}, \tilde{x}) \in \mathbb{R}^n \times \mathbb{R}^k : |\tilde{x} - \tilde{a}| \leq \gamma \rho\},$$

$$(1.3) \quad \rho^{-n} \mu_{t_1}(\mathbf{B}(a, \rho)) \leq (2 - \lambda) \omega_n$$

and set $I := (t_1 + \gamma^{\alpha_0} \rho^2, t_2)$.

Then there exists an $u \in \mathcal{C}^\infty(I \times \mathbf{B}^n(\hat{a}, \gamma_0 \rho), \mathbb{R}^k)$ such that

$$\mu_t \llcorner \mathbf{C}(a, \gamma_0 \rho, \rho) = \mathcal{H}^n \llcorner \text{graph}(u(t, \cdot)) \quad \text{for all } t \in I.$$

Moreover $\sup |Du(t, \cdot)| \leq \alpha_0^{-1} \rho^{-2} (t - t_1)$ for all $t \in I$ and $F_t(\hat{x}) = (\hat{x}, u(t, \hat{x}))$ satisfies (1.1).

The main difference to previous versions is that here we obtain regularity up to the time t_2 at which we assumed the non-vanishing, were in Brakke's original theorem measure bounds from below have to be assumed for later times.

We also obtain a local regularity theorem similar to the one of White [Whi05], see also Ecker's version [Eck04, Thm. 5.6]. We show that a non-vanishing Brakke flow which locally has Gaussian density ratios close to one will become graphical in a small neighbourhood.

1.2 Theorem. *For every $\beta \in (0, 1)$ there exists an $\eta \in (0, 1)$ such that the following holds:*

Let $\rho \in (0, \infty)$, $\rho_0 \in [\rho, \infty)$, $t_0 \in \mathbb{R}$, $a \in \mathbb{R}^{n+k}$ and let $(\mu_t)_{t \in [t_0 - \rho^2, t_0]}$ be a Brakke flow in $\mathbf{B}(a, 4\sqrt{n}\rho_0)$. Suppose $a \in \text{spt} \mu_{t_0}$ and for all $(s, y) \in (t_0 - \rho^2, t_0] \times \mathbf{B}(a, \rho)$ we have

$$(1.4) \quad \int_{\mathbb{R}^{n+k}} \Phi_{(s,y)} \varphi_{(s,y), \rho_0} d\mu_{t_0 - \rho^2} \leq 1 + \eta,$$

where Φ and φ are from Definition 2.7. Set $I := (t_0 - \eta^2 \rho^2, t_0)$.

Then there exist $S \in \mathbf{O}(n+k)$ and $u \in \mathcal{C}^\infty(I \times \mathbf{B}^n(0, \eta\rho), \mathbb{R}^{n+k})$ such that for $M_t = \text{graph}(u(t, \cdot))$ we have

$$\mu_t \llcorner \mathbf{B}(a, \eta\rho) = \mathcal{H}^n \llcorner (S[M_t] + a \cap \mathbf{B}(a, \eta\rho)) \quad \text{for all } t \in I.$$

Moreover $\sup |Du| \leq \beta$ and $F_t(\hat{x}) = (\hat{x}, u(t, \hat{x}))$ satisfies (1.1).

One key ingredient in order to obtain these regularity results is the observation that a non-vanishing Brakke flow which is initially graphical with small gradient will stay graphical for some time. This is the non-smooth version of the corresponding theorem by Ilmanen, Neves and Schulze [INS14, Thm. 1.5].

1.3 Theorem. *There exists an $l_0 \in (0, 1)$ such that the following holds:*

Let $l \in [0, l_0]$, $\rho \in (0, \infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1, t_1 + l_0 \rho^2]$, $a = (\hat{a}, \tilde{a}) \in \mathbb{R}^n \times \mathbb{R}^k$ and let $(\mu_t)_{t \in [t_1, t_2]}$ be a Brakke flow in $\mathbf{C}(a, 2\rho, 2\rho)$. Assume $a \in \text{spt} \mu_{t_1}$ and

$$(1.5) \quad \text{spt} \mu_{t_2} \cap \mathbf{C}(a, \rho, \rho) \neq \emptyset.$$

Suppose there exists an $u_0 \in \mathcal{C}^{0,1}(\mathbf{B}^n(\hat{a}, 2\rho), \mathbb{R}^k)$ with $\text{lip}(u_0) \leq l$ and

$$(1.6) \quad \mu_{t_1} \llcorner \mathbf{C}(a, 2\rho, 2\rho) = \mathcal{H}^n \llcorner \text{graph}(u_0).$$

Then there exists an $u \in \mathcal{C}^\infty((t_1, t_2) \times \mathbf{B}^n(\hat{a}, \rho), \mathbb{R}^k)$ such that

$$\mu_t \llcorner \mathbf{C}(a, \rho, \rho) = \mathcal{H}^n \llcorner \text{graph}(u(t, \cdot)) \quad \text{for all } t \in (t_1, t_2).$$

Moreover $\sup |Du(t, \cdot)| \leq \sqrt[4]{l} + \rho^{-2}(t - t_1)$ for all $t \in (t_1, t_2)$ and $F_t(\hat{x}) = (\hat{x}, u(t, \hat{x}))$ satisfies (1.1).

1.4 Remark. In all the above results $F_t(\hat{x}) = (\hat{x}, u(t, \hat{x}))$ satisfies (1.1) and $|Du|$ is small, thus the results for smooth graphical mean curvature flow can be applied to obtain bounds on the curvature of $\text{graph}(u(t, \cdot))$ and higher derivatives of it, see Appendix A, in particular Proposition A.4 and Lemma A.6. This also yields estimates on the higher derivatives of u .

Note that in the above results we cannot expect to obtain a graphical representation at the final time, see Example 2.5.

To deal with the potential vanishing of Brakke flows we also use some sort of continuation result for varifolds. We show that a unit density varifold with absolutely continuous first variation that is contained in a Lipschitz graph is either empty or coincides with the graph. This generalizes Allard's constancy theorem [All72, Thm. 4.6.(3)] (see also Simon [Sim83, Thm. 8.4.1]) to Lipschitz graphs, but additionally requires unit density.

1.5 Theorem. Let $D \subset \mathbb{R}^n$ be open and connected with $(n-1)$ -rectifiable boundary $\partial D = \bar{D} \setminus D$ and set $U := D \times \mathbb{R}^k$. Consider a unit density n -rectifiable Radon measure μ and a Lipschitz function $f : D \rightarrow \mathbb{R}^k$ such that

$$(1.7) \quad \emptyset \neq \text{spt} \mu \cap U \subset \text{graph} f$$

$$(1.8) \quad \mu(A) = 0 \text{ implies } \|\delta \mu\|(A) = 0 \text{ for all } A \subset U.$$

Then $\mu \llcorner U = \mathcal{H}^n \llcorner \text{graph} f$.

Note that Duggan [Dug86, Thm. 4.5] shows constant density for stationary n -rectifiable Radon measures which already equal a Lipschitz graph.

Organisation and sketch of the proof We start by recalling some definitions and important results in Section 2.

Then in Section 3 we show Theorem 1.5. In the proof we employ the Gauss–Green theorem by Federer [Fed69, §4.5.6] to see that the projection of $\text{spt} \mu$ onto \mathbb{R}^n is stationary in D and subsequently the result follows from Allard's constancy theorem [All72, Thm. 4.6.(3)].

Section 4 establishes that a local height bound is maintained under Brakke flow, see Proposition 4.1. Here we improve the height bound increases in time from the known linear growth to arbitrary high exponents.

The main part of this work is Section 5, where Theorem 1.3 is established. Essentially we consider a Brakke flow in $\mathbf{C}(0, 2, 2)$ for times in $[0, \tau]$ such that $\text{spt} \mu_\tau \cap \mathbf{C}(0, \delta, 1) \neq \emptyset$. First assume as initial condition that $\mu_0 \llcorner \mathbf{C}(0, 2, 2)$ lies in a slab of height h and satisfies certain density ratio assumptions. Based on Brakke's local regularity theorem [Bra78, Thm. 6.11] and the height bound

from Section 4 we show that the flow is graphical inside $\mathbf{C}(0, h, 1)$ for times in $[Ch^2, \tau - Ch^2]$ if h is small enough, $\delta \leq h$ and $\tau \leq \sqrt[32]{h}$. Under stronger density assumptions we actually obtain graphical representability inside $\mathbf{C}(0, 1, 1)$ for times in $[Ch, \tau - Ch]$, see Lemma 5.2.

Now Change the initial condition to $\mu_0 \llcorner \mathbf{C}(0, 2, 2)$ being graphical with Lipschitz constant smaller than l . This allows applying Lemma 5.2 on arbitrary small scales, which yields that the flow is graphical inside $\mathbf{C}(0, 1, 1)$ for times in $[0, \tau - Cl]$ if l is small enough, $\delta \leq l$ and $\tau \leq \sqrt[32]{l}$. Iterating this result leads to Lemma 5.4 which says that the flow is graphical inside $\mathbf{B}(0, L\delta)$ for times in $[0, \tau - \delta^2]$ if we choose l small enough depending on L and suppose $\delta \leq l, \tau \leq l$. Here the curvature estimates from Appendix A for smooth mean curvature flow are crucial to control the gradient during the iteration. Using Lemma 5.4 with varying center points and arbitrary small δ we obtain that $\text{spt}\mu_t \cap \mathbf{C}(0, 1, 1)$ is contained in a Lipschitz graph and has unit density for almost all $t \in [0, \tau]$. In view of Theorem 1.5 this lets us conclude Theorem 1.3.

Section 6 contains the proof of Theorem 1.1. First we see that Theorem 1.3 and Lemma 5.2 directly imply a version of Theorem 1.1, which assumes stronger density bounds initially, see Lemma 6.1. Then we use Brakke's cylindrical growth theorem [Bra78, Lem. 6.4] to simplify these assumptions, which establishes Theorem 1.1 in the desired form.

In Section 7 Theorem 1.2 is proven. In order to do so we first employ Huisken's monotonicity formula [Hui90, Thm. 3.1] to show that non-moving planes are the only Brakke flows in \mathbf{R}^{n+k} that have Gaussian density ratios bounded everywhere by 1. Then under the assumptions of Theorem 1.2 a blow up argument and Ilmanen's compactness theorem yield that in a small neighbourhood the conditions of Theorem 1.1 are satisfied, which yields the conclusion of Theorem 1.2.

In Appendix A we show some curvature estimates for smooth mean curvature flow. Most of the presented results are already known despite small variations. Finally Appendix B contains some remarks.

Acknowledgements. I want to thank Ulrich Menne for his help and advice, in particular for the proof of Theorem 1.5. Also I want to thank Matthew Langford and Yoshihiro Tonegawa for some helpful discussions.

2 Preliminaries

Notation. For an excellent introduction to geometric measure theory we recommend the lecture notes by Simon [Sim83]. Here we recall the most important definitions.

- We set $\mathbb{R}^+ := \{x \in \mathbb{R}, x \geq 0\}$, $\mathbb{N} := \{1, 2, 3, \dots\}$ and $(a)_+ := \max\{a, 0\}$ for $a \in \mathbb{R}$.
- We fix $\mathbf{n}, \mathbf{k} \in \mathbb{N}$. Quantities that only depend on \mathbf{n} and/or \mathbf{k} are considered constant. Such a constant may be denoted by C or c , in particular the value of C and c may change in each line.
- For $a \in \mathbb{R}^{n+k}$ the values $\hat{a} \in \mathbb{R}^n$ and $\tilde{a} \in \mathbb{R}^k$ are given by $a = (\hat{a}, \tilde{a})$.

- We denote the canonical basis of \mathbb{R}^{n+k} by $(\mathbf{e}_i)_{1 \leq i \leq n+k}$. In particular the canonical basis of \mathbb{R}^n is $(\hat{\mathbf{e}}_i)_{1 \leq i \leq n}$.

Consider $n, k \in \mathbb{N}$.

- Let $\mathbf{O}(n)$ denote the space of rotations on \mathbb{R}^n . Let $\mathbf{G}(n+k, n)$ denote the space of n -dimensional subspaces of \mathbb{R}^{n+k} . For $T \in \mathbf{G}(n+k, n)$ set $T^\perp := \{x \in \mathbb{R}^{n+k} : x \cdot v = 0 \quad \forall v \in T\}$. By $T_\sharp : \mathbb{R}^{n+k} \rightarrow T$ we denote the projection onto T .
- For $R, r, h \in (0, \infty)$ and $a \in \mathbb{R}^n$ we set

$$\begin{aligned} \mathbf{B}^n(a, R) &:= \{x \in \mathbb{R}^n : |x - a| < R\}, \quad \mathbf{B}(a, R) := \mathbf{B}^{n+k}(a, R), \\ \mathbf{C}(a, r, h) &:= \mathbf{B}^n(\hat{a}, r) \times \mathbf{B}^k(\tilde{a}, h), \quad \mathbf{C}(a, r) := \mathbf{B}^n(\hat{a}, r) \times \mathbb{R}^k. \end{aligned}$$

- Consider open sets $I \subset \mathbb{R}$ and $V \subset \mathbb{R}^n$. For $f \in \mathcal{C}^1(I \times V)$ we denote by $\partial_t f$ the derivative of f in I , while Df denotes the derivative of f in V . If $(\mu_t)_{t \in I}$ is a family of Radon measures on V we often abbreviate $\int_V f(t, x) d\mu_t(x) = \int_V f d\mu_t$.
- Let \mathcal{L}^n denote the n -dimensional Lebesgue measure and \mathcal{H}^n denote the n -dimensional Hausdorff measure. Set $\omega_n := \mathcal{L}^n(B^n(0, 1))$.

Consider an open set $U \subset \mathbb{R}^{n+k}$ be open and a Radon measure μ on U .

- Set $\text{spt} \mu := \{x \in U : \mu(\mathbf{B}^{n+k}(x, r)) > 0, \text{ for all } r \in (0, \infty)\}$.
- Consider $x \in U$. We define the upper and lower density by

$$\Theta^{*n}(\mu, x) := \limsup_{r \searrow 0} \frac{\mu(\mathbf{B}^{n+k}(x, r))}{\omega_n r^n}, \quad \Theta_*^n(\mu, x) := \liminf_{r \searrow 0} \frac{\mu(\mathbf{B}^{n+k}(x, r))}{\omega_n r^n}$$

and if both coincide the value is denoted by $\Theta^n(\mu, x)$ and called the density of μ at x .

- Consider $y \in U$. If there exist $\theta(y) \in (0, \infty)$ and $\mathbf{T}(\mu, y) \in \mathbf{G}(n+k, n)$ such that

$$\lim_{\lambda \searrow 0} \lambda^{-n} \int_U \phi(\lambda^{-1}(x - y)) d\mu(x) = \theta(y) \int_{\mathbf{T}(\mu, y)} \phi(x) d\mathcal{H}^n(x)$$

for all $\phi \in \mathcal{C}_c^0(\mathbb{R}^{n+k})$, then $\mathbf{T}(\mu, y)$ is called the (n -dimensional) approximate tangent space of μ at x with multiplicity $\theta(y)$.

- We say μ is n -rectifiable, if the approximate tangent space exists at μ -a.e. $x \in U$. Note that in this case $\theta(x) = \Theta^n(\mu, x)$ for μ -a.e. $x \in U$. We say μ is integer n -rectifiable, if μ is n -rectifiable and $\Theta^n(\mu, x) \in \mathbb{N}$ for μ -a.e. $x \in U$. We say μ has unit density, if μ is n -rectifiable and $\Theta^n(\mu, x) = 1$ for μ -a.e. $x \in U$.

Let μ be an n -rectifiable Radon measure on U

- Consider $\phi \in \mathcal{C}^1(U, \mathbb{R}^{n+k})$. For $x \in U$ such that $\mathbf{T}(\mu, x)$ exists set $\text{div}_\mu \phi(x) := \sum_{i=1}^n (D_{b_i} \phi(x)) \cdot b_i$, where $(b_i)_{1 \leq i \leq n}$ is an orthonormal basis of $\mathbf{T}(\mu, x)$.

- Denote the first variation of μ in U by $\delta\mu(\phi) := \int_U \operatorname{div}_\mu \phi \, d\mu$ for $\phi \in \mathcal{C}_c^1(U, \mathbb{R}^{n+k})$. Set $\|\delta\mu\|(A) := \sup\{\partial\mu(\phi), \phi \in \mathcal{C}_c^1(A, \mathbb{R}^{n+k}), |\phi| \leq 1\}$ for $A \subset U$ open.
- If there exists $\mathbf{H}_\mu : \operatorname{spt}\mu \rightarrow \mathbb{R}^{n+k}$ such that \mathbf{H}_μ is locally μ -integrable and $\delta\mu(\phi) = \int_U \mathbf{H}_\mu \cdot \phi \, d\mu$ for all $\phi \in \mathcal{C}_c^1(U, \mathbb{R}^{n+k})$, then \mathbf{H}_μ is called the generalised mean curvature vector of μ in U .

Brakke flow. An introduction to the Brakke flow can be found in [Bra78], [Ilm94], [KT14] or [Lah14].

2.1 Definition. For a Radon measure μ on \mathbb{R}^{n+k} and a $\phi \in \mathcal{C}_c^1(\mathbb{R}^{n+k})$ we define the Brakke variation $\mathcal{B}(\mu, \phi)$ as follows: If $\mu \llcorner \{\phi > 0\}$ is \mathbf{n} -rectifiable, has generalised mean curvature vector \mathbf{H}_μ in $\{\phi > 0\}$ and $\int_{\{\phi > 0\}} |\mathbf{H}_\mu|^2 \, d\mu < \infty$ then set

$$\mathcal{B}(\mu, \phi) := \int_{\mathbb{R}^{n+k}} ((\mathbf{T}(\mu, x)^\perp)_\sharp D\phi(x) \cdot \mathbf{H}_\mu(x) - \phi(x) |\mathbf{H}_\mu(x)|^2) \, d\mu(x).$$

Else we set $\mathcal{B}(\mu, \phi) := -\infty$. Note that in case μ is integer \mathbf{n} -rectifiable, by a deep theorem of Brakke [Bra78, Thm. 5.8], we have $\mathbf{H}_\mu(x) \perp \mathbf{T}(\mu, x)$ for μ -a.e. $x \in \mathbb{R}^{n+k}$. Hence in this case the projection can be left out.

2.2 Remark ([Bra78, Prop. 3.4], [Ilm94, Lem. 6.6]). If $\phi \in \mathcal{C}_c^2(\mathbb{R}^{n+k})$ and $\mathcal{B}(\mu, \phi) > -\infty$ we can estimate

$$\mathcal{B}(\mu, \phi) \leq \sup |D^2\phi| \, \mu(\{\phi > 0\}) - \frac{1}{2} \int_{\mathbb{R}^{n+k}} |\mathbf{H}_\mu|^2 \phi \, d\mu.$$

2.3 Definition. Let $U \subset \mathbb{R}^{n+k}$ be open, $t_1 \in \mathbb{R}$, $t_2 \in (t_1, \infty)$ and $(\mu_t)_{t \in [t_1, t_2]}$ be a family of radon measures on \mathbb{R}^{n+k} . We call $(\mu_t)_{t \in [t_1, t_2]}$ a Brakke flow in U if $\mu_t \llcorner U$ is integer \mathbf{n} -rectifiable for a.e. $t \in (t_1, t_2)$ and for all $t_1 \leq s_1 < s_2 \leq t_2$ we have

$$(2.1) \quad \mu_{s_2}(\phi(s_2, \cdot)) - \mu_{s_1}(\phi(s_1, \cdot)) \leq \int_{s_1}^{s_2} (\mathcal{B}(\mu_t, \phi(t, \cdot)) + \mu_t(\partial_t \phi(t, \cdot))) \, dt$$

for all $\phi \in \mathcal{C}^1((s_1, s_2) \times U) \cap \mathcal{C}^0([s_1, s_2] \times U)$ with $\cup_{t \in [s_1, s_2]} \operatorname{spt}\phi(t, \cdot) \subset \subset U$.

2.4 Remark. Suppose $(\mu_t)_{t \in [t_1, t_2]}$ is a Brakke flow in U then we have:

- For a.e. $t \in (t_1, t_2)$ we have: $\mu_t \llcorner U$ is integer \mathbf{n} -rectifiable, has generalised mean curvature vector \mathbf{H}_{μ_t} in U and $\int_K |\mathbf{H}_{\mu_t}|^2 \, d\mu_t < \infty$ for all $K \subset \subset U$.
- For $(s_0, y_0) \in \mathbb{R} \times \mathbb{R}^{n+k}$ and $r \in (0, \infty)$ set $\nu_t(A) := r^{-n} \mu_{r^2 t + s_0}(rA + y_0)$, then $(\nu_t)_{t \in [r^{-2}(t_1 - s_0), r^{-2}(t_2 - s_0)]}$ is a Brakke flow in $r^{-1}(U - y_0)$.

The Brakke flow allows the sudden loss of mass. In particular we have

2.5 Example. For $0 < t_0 \leq T$ and $0 < \epsilon < \rho < \infty$ consider the Brakke flow $(\mu_t)_{t \in [0, T]}$ given by $\mu_t = \mathcal{H}^n \llcorner (\mathbb{R}^n \times \{0\}^k)$ for $t \in [0, t_0)$, $\mu_{t_0} = \mathcal{H}^n \llcorner (\mathbf{B}^n(0, \epsilon) \times \{0\}^k)$ and $\mu_t := \emptyset$ for $t \in (t_0, T]$. Note that μ_t is graphical with Lipschitz constant 0 for $t \in [0, t_0)$ and $0 \in \operatorname{spt}\mu_{t_0}$ but $\mu_{t_0} \llcorner \mathbf{B}(0, \rho)$ is not graphical.

Important results Here we recall some important results that are crucial for the proofs in this article.

2.6 Lemma (Measure bound [Bra78, Thm. 3.7],[Eck04, Prop. 4.9]). *Let $R \in (0, \infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1, t_1 + (2\mathbf{n})^{-1}R^2]$, $z_0 \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and let $(\mu_t)_{t \in [t_1, t_2]}$ be a Brakke flow in $\mathbf{B}(z_0, 2R)$.*

Then for all $t \in [t_1, t_2]$ we have

$$\mu_t(\mathbf{B}(z_0, R)) \leq 8\mu_{t_1}(\mathbf{B}(z_0, 2R)).$$

2.7 Definition. Let $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and $\rho \in (0, \infty)$ be fixed. Set

$$\begin{aligned}\Phi_{(t_0, x_0)}(t, x) &:= (4\pi(t_0 - t))^{-\frac{\mathbf{n}}{2}} \exp\left(\frac{|x - x_0|^2}{4(t - t_0)}\right). \\ \varphi_{(t_0, x_0), \rho}(t, x) &:= (1 - \rho^{-2}(|x - x_0|^2 + 2\mathbf{n}(t - t_0)))_+^3\end{aligned}$$

for $(t, x) \in \mathbb{R} \times \mathbb{R}^{\mathbf{n}+\mathbf{k}}$, where $\Phi_{(t_0, x_0)}(t, x)$ is only defined for $t \leq t_0$.

2.8 Theorem (Monotonicity formula [Hui90, Thm. 3.1],[Ilm95, Thm. 7][Eck04, Thm. 4.13]). *Consider an open set $U \subset \mathbb{R}^{\mathbf{n}+\mathbf{k}}$, $\rho, D \in (0, \infty)$, $(t_0, x_0) \in \mathbb{R} \times U$, $s_1 \in (-\infty, t_0)$ and $s_2 \in (s_1, t_0)$ and let $(\mu_t)_{t \in [s_1, s_2]}$ be a Brakke flow in U . Assume one (or both) of the following holds*

1. $\text{spt}\varphi_{(t_0, x_0), \rho}(s_1, \cdot) \subset\subset U$.
2. $U = \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and $\sup_{t \in [s_1, s_2]} \sup_{R \in (0, \infty)} \mu_t(\mathbf{B}(x_0, R)) \leq DR^{\mathbf{n}}$.

Then

$$\begin{aligned}& \int_U \Phi(s_2, x) d\mu_{s_2}(x) - \int_U \Phi(s_1, x) d\mu_{s_1}(x) \\ & \leq \int_{s_1}^{s_2} \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \left(\Phi(t, x) \left| \mathbf{H}_{\mu_t}(x) + \frac{(\mathbf{T}(\mu_t, x)^\perp)_\sharp(x - x_0)}{2(t_0 - t)} \right|^2 \right) d\mu_t(x) dt\end{aligned}$$

for $\Phi = \Phi_{(t_0, x_0)}\varphi_{(t_0, x_0), \rho}$ if assumption 1 holds and $\Phi = \Phi_{(t_0, x_0)}$ if assumption 2 holds. Here the term under the time integral is interpreted as $-\infty$ at times where one of the technical conditions fails, as in Definition 2.1.

2.9 Lemma (Clearing out [Bra78, Lem. 6.3]). *There exist $C \in (1, \infty)$ and $\alpha_1 := (\mathbf{n} + 6)^{-1}$ such that the following holds:*

Let $\eta \in [0, \infty)$, $R \in (0, \infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1 + C\eta^{2\alpha_1}R^2, t_1 + (4\mathbf{n})^{-1}R^2)$, $x_0 \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$. Let $U \subset \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ be open with $U \supset\supset \mathbf{B}(x_0, R)$ and let $(\mu_t)_{t \in [t_1, t_2]}$ be a Brakke flow in $\mathbf{B}(x_0, R)$. Suppose we have

$$R^{-\mathbf{n}} \int_U (1 - R^{-2}|x - x_0|^2)_+^3 d\mu_{t_1} \leq \eta.$$

Set $R(t) := \sqrt{R^2 - 4\mathbf{n}(t - t_0)}$.

Then for all $t \in [t_1 + C\eta^{2\alpha_1}R^2, t_2]$ we have $\mu_t(\mathbf{B}(x_0, R(t))) = 0$.

2.10 Theorem (Local regularity [Bra78, Thm. 6.11], [KT14, Thm. 8.7], [Lah14, Thm. 8.4]). *For every $\lambda \in (0, 1]$ there exist $\Lambda \in (1, \infty)$ and $h_0 \in (0, 1)$ such that the following holds:*

Let $h \in (0, h_0]$, $R \in (0, \infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1 + 2\Lambda R^2, \infty)$, $x_0 \in \mathbb{R}^{n+k}$, and let $(\mu_t)_{t \in [t_1, t_2]}$ be a Brakke flow in $\mathbf{B}(x_0, 4R)$. Suppose we have

$$(2.2) \quad \text{spt} \mu_t \cap \mathbf{B}(x_0, 4R) \subset \mathbf{C}(x_0, 4R, hR)$$

$$(2.3) \quad R^{-n} \mu_t (\mathbf{B}(x_0, 4R)) \leq \lambda^{-1}$$

for all $t \in [t_1, t_2]$ and

$$(2.4) \quad R^{-n} \mu_{t_1} (\mathbf{B}(x_0, R)) \leq (2 - \lambda) \omega_n$$

$$(2.5) \quad R^{-n} \mu_{t_2} (\mathbf{B}(x_0, R/2)) \geq \lambda \omega_n.$$

Set $I := (t_1 + \Lambda R^2, t_2 - \Lambda R^2)$.

Then there exists an $u \in C^\infty(I \times \mathbf{B}^n(\hat{x}_0, h_0 R), \mathbb{R}^k)$ such that

$$\mu_t \llcorner \mathbf{C}(x_0, h_0 R, R) = \mathcal{H}^n \llcorner \text{graph}(u(t, \cdot)) \quad \text{for all } t \in I.$$

Moreover $\sup |Du| \leq \Lambda h$ and $F_t(\hat{x}) = (\hat{x}, u(t, \hat{x}))$ satisfies (1.1).

To deduce this result from [Bra78, Thm. 6.11], [KT14, Thm. 8.7] you also need to use [Bra78, Lem. 6.6], [KT14, Thm. 5.7] to see that the density ratio bounds (2.4) and (2.5) actually hold at all times in slightly weaker form. Note that Brakke as well as Kasai and Tonegawa state this theorem for unit density Brakke flows, though their proofs only use integer density. For the smoothness of u we refer to [Ton15, Thm. 3.6].

2.11 Theorem (Compactness [Ilm94, Thm. 7.1]). *Let $t_1 \in \mathbb{R}$ and $t_2 \in (t_1, \infty)$. For all $i \in \mathbb{N}$ consider an open set $U_i \subset \mathbb{R}^{n+k}$ and a Brakke flow $(\mu_t^i)_{t \in [t_1, t_2]}$ in U_i . Assume $U_i \subset U_{i+1}$ for all $i \in \mathbb{N}$ and set $U := \bigcup_{i=1}^\infty U_i$. Suppose for every $K \subset\subset U$ there exists an $C_K \in (1, \infty)$ such that*

$$\sup_{i \in \mathbb{N}} \sup_{t \in [t_1, t_2]} \mu_t^i(K \cap U_i) \leq C_K.$$

Then there exists a subsequence $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ and a Brakke flow $(\mu_t)_{t \in [t_1, t_2]}$ in U such that

$$\mu_t(\phi) = \lim_{j \rightarrow \infty} \mu_t^{\sigma(j)}(\phi) \quad \text{for all } \phi \in C_c^0(U_{j_0})$$

for all $t \in [t_1, t_2]$ and all $j_0 \in \mathbb{N}$.

Actually in [Ilm94] Ilmanen assumes $U_i \equiv M$, for a complete manifold M . To derive the above result from [Ilm94, Thm. 7.1] use a diagonal subsequence argument, see Remark B.3 for some more details.

2.12 Lemma (Tilt-bound [Bra78, Lem. 5.5]). *There exists a $C \in (0, \infty)$ such that the following holds:*

Let $U \subset \mathbb{R}^{n+k}$ be open and let μ be an integer \mathbf{n} -rectifiable Radon measure on U with locally \mathcal{L}^2 -integrable mean curvature vector \mathbf{H}_μ . Consider $g \in C_c^1(U, \mathbb{R})$,

$f, h \in C_c^0(U, \mathbb{R})$ with $g^2 \leq fh$ and set

$$\begin{aligned}\alpha_f^2 &:= \int_U |\mathbf{H}_\mu(x)|^2 f(x)^2 d\mu(x), \\ \beta_g^2 &:= \int_U \|(\mathbb{R}^n \times \{0\}^{\mathbf{k}})_\sharp - \mathbf{T}(\mu, x)_\sharp\|^2 g(x)^2 d\mu(x), \\ \gamma_h^2 &:= \int_U |\tilde{x}|^2 h(x)^2 d\mu(x), \\ \xi_g^2 &:= \int_U |\tilde{x}|^2 |\nabla^\mu g(x)|^2 d\mu(x).\end{aligned}$$

Then we have $\beta_g^2 \leq C(\alpha_f \gamma_h + \xi_g^2)$.

2.13 Theorem (Cylindrical growth [Bra78, Thm. 6.4]). *Let $U \subset \mathbb{R}^{n+k}$ be open, $R_1 \in (0, \infty)$, $R_2 \in (R_1, \infty)$, $\alpha, \beta \in [0, \infty)$. Let μ be an integer \mathbf{n} -rectifiable Radon measure on U with \mathcal{L}^2 -integrable mean curvature vector \mathbf{H}_μ and $\text{spt}\mu \cap \mathbf{C}(x_0, R_2) \subset\subset U$. Consider $\psi \in C_c^3([-1, 1], \mathbb{R}^+)$. Suppose for all $r \in [R_1, R_2]$ we can estimate*

$$(2.6) \quad r^{-n} \int_U |\mathbf{H}_\mu(x)|^2 \psi(r^{-1}|\hat{x}|) d\mu(x) \leq \alpha^2,$$

$$(2.7) \quad r^{-n} \int_U \|(\mathbb{R}^n \times \{0\}^{\mathbf{k}})_\sharp - \mathbf{T}(\mu, x)_\sharp\|^2 \psi(r^{-1}|\hat{x}|) d\mu(x) \leq \beta^2.$$

Then we have

$$\begin{aligned}& \left| R_2^{-n} \int_U \psi(R_2^{-1}|\hat{x}|) d\mu(x) - R_1^{-n} \int_U \psi(R_1^{-1}|\hat{x}|) d\mu(x) \right| \\ & \leq n \log(R_2/R_1) \beta^2 + (R_2 - R_1) \alpha \beta + \beta^2.\end{aligned}$$

3 Graphs without holes

In this section we prove Theorem 1.5. Consider a unit density Radon measure μ such that the first variation $\delta\mu$ is absolutely continuous with respect to μ . In some sense this should imply that μ has no ‘boundary points’. Here we show that, if such a μ is contained in the graph of some Lipschitz function f , then μ actually coincides with the measure generated by the graph of f . For $f \in C^2$ and stationary μ this is a direct consequence of Allard’s constancy theorem [All72, Thm. 4.6.(3)] (see also Simon’s notes [Sim83, Thm. 8.4.1]). Here we use the generalized Gauss-Green theorem by Federer [Fed69, §4.5.6] to show that the projection of μ onto $\mathbb{R}^n \times \{0\}^{\mathbf{k}}$ is stationary, which reduces our problem to the C^2 -setting, thus implying the result.

3.1 Definition. Let μ be an n -rectifiable Radon measure on \mathbb{R}^{n+k} . We denote the associated general varifold by $\mathbf{V}(\mu)$, i.e. $\mathbf{V}(\mu)$ is the Radon measure on $\mathbb{R}^{n+k} \times \mathbf{G}(n+k, n)$ given by

$$\mathbf{V}(\mu)(A) := \mu(\{x \in \mathbb{R}^{n+k} : (x, \mathbf{T}(\mu, x)) \in A\}) \quad \text{for } A \subset \mathbb{R}^{n+k}.$$

For $y \in \mathbb{R}^{n+k}$ and $\lambda \in (0, \infty)$ we define the λ -blow-up around y by

$$\mu_{y, \lambda}(A) := \lambda^{-n} \mu(\lambda A + y) \quad \text{for } A \subset \mathbb{R}^{n+k}.$$

Proof of Theorem 1.5. This proof is based on ideas by Ulrich Menne. Define $F : D \rightarrow U$ by $F(\hat{x}) := (\hat{x}, f(\hat{x}))$. Set

$$\begin{aligned} U_1 &:= \{x \in U : \Theta^{\mathbf{n}-1}(\|\delta\mu\|, x) = 0\}, \\ Q_1 &:= \{x \in U : \Theta_*^{\mathbf{n}}(\mu, x) \geq 1\}, \quad Q_2 := Q_1 \cap U_1, \\ R_1 &:= \{x \in U : \Theta^{\mathbf{n}}(\mu, x) = 0\}, \quad R_2 := R_1 \cap U_1. \end{aligned}$$

We claim

$$(3.1) \quad \mathcal{H}^{\mathbf{n}-1}(U \setminus (Q_2 \cup R_2)) = 0.$$

As we have absolutely continuous first variation we can use a result by Menne [Men09, Rem. 2.11] to see $\mathcal{H}^{\mathbf{n}-1}(U \setminus (Q_1 \cup R_1)) = 0$. Hence, to establish the claim it remains to show

$$(3.2) \quad \mathcal{H}^{\mathbf{n}-1}(U \setminus U_1) = 0.$$

We proceed as Federer and Ziemer [FZ72, §8]. For $i \in \mathbb{N}$ set

$$B_i := \{x \in U \cap \mathbf{B}(0, i) : \Theta^{*\mathbf{n}-1}(\|\delta\mu\|, x) > i^{-1}\}.$$

Then by Federer [Fed69, §2.10.19(3)] we have $i\|\partial\mu\|(B_i) \geq \mathcal{H}^{\mathbf{n}-1}(B_i)$ for all $i \in \mathbb{N}$. This and (1.8) yield the following chain of implications: B_i bounded, $\|\partial\mu\|(B_i) < \infty$, $\mathcal{H}^{\mathbf{n}-1}(B_i) < \infty$, $\mathcal{H}^{\mathbf{n}}(B_i) = 0$, $\mu(B_i) = 0$, $\|\partial\mu\|(B_i) = 0$, $\mathcal{H}^{\mathbf{n}-1}(B_i) = 0$. This shows (3.2) which completes the proof of (3.1).

Now set

$$\begin{aligned} p &:= (\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}})_{\mathbf{t}}, \quad A_0 := p[\text{spt}\mu] \cap D, \\ Q_0 &:= \{\hat{x} \in \mathbb{R}^{\mathbf{n}} : \Theta^{\mathbf{n}}(\mathcal{L}^{\mathbf{n}} \llcorner (\mathbb{R}^{\mathbf{n}} \setminus A_0), \hat{x}) = 0\}, \\ R_0 &:= \{\hat{x} \in \mathbb{R}^{\mathbf{n}} : \Theta^{\mathbf{n}}(\mathcal{L}^{\mathbf{n}} \llcorner A_0, \hat{x}) = 0\}. \end{aligned}$$

We want to use

$$(3.3) \quad p[Q_2] \subset Q_0 \text{ and } p[R_2 \cap \text{graph}f] \subset R_0.$$

We will prove this statement later. Suppose (3.3) holds, then

$$\begin{aligned} \mathcal{H}^{\mathbf{n}-1}(D \setminus (Q_0 \cup R_0)) &\leq \mathcal{H}^{\mathbf{n}-1}(F[D \setminus (Q_0 \cup R_0)]) \\ &\leq \mathcal{H}^{\mathbf{n}-1}(F[D \setminus (p[Q_2] \cup p[R_2 \cap \text{graph}f])]) = \mathcal{H}^{\mathbf{n}-1}(\text{graph}f \setminus (Q_2 \cup R_2)). \end{aligned}$$

Hence by (3.1) we have

$$(3.4) \quad \mathcal{H}^{\mathbf{n}-1}(D \setminus (Q_0 \cup R_0)) = 0.$$

We say $\hat{v} \in \partial\mathbf{B}^{\mathbf{n}}(0, 1)$ is an external normal of A_0 at $\hat{y} \in \mathbb{R}^{\mathbf{n}}$, if

$$\begin{aligned} \Theta^{\mathbf{n}}(\mathcal{L}^{\mathbf{n}} \llcorner \{\hat{x} \in \mathbb{R}^{\mathbf{n}} : (\hat{x} - \hat{y}) \cdot \hat{v} > 0\} \cap A_0, \hat{y}) &= 0 \\ \text{and } \Theta^{\mathbf{n}}(\mathcal{L}^{\mathbf{n}} \llcorner \{\hat{x} \in \mathbb{R}^{\mathbf{n}} : (\hat{x} - \hat{y}) \cdot \hat{v} < 0\} \setminus A_0, \hat{y}) &= 0, \end{aligned}$$

Let B_0 be the set consisting of all $\hat{y} \in \mathbb{R}^{\mathbf{n}}$ for which there exists an external normal of A_0 at \hat{y} . Then we have

$$(3.5) \quad B_0 \cap (Q_0 \cup R_0) = \emptyset.$$

To see this consider $\hat{y} \in Q_0$ and $\hat{v} \in \partial \mathbf{B}^{\mathbf{n}}(0, 1)$. We can estimate

$$\begin{aligned} & \mathcal{L}^{\mathbf{n}}(\{\hat{x} \in \mathbb{R}^{\mathbf{n}} : (\hat{x} - \hat{y}) \cdot \hat{v} > 0\} \cap A_0 \cap \mathbf{B}^{\mathbf{n}}(\hat{y}, r)) \\ & \geq \mathcal{L}^{\mathbf{n}}(\{\hat{x} \in \mathbb{R}^{\mathbf{n}} : (\hat{x} - \hat{y}) \cdot \hat{v} > 0\} \cap \mathbf{B}^{\mathbf{n}}(\hat{y}, r)) - \mathcal{L}^{\mathbf{n}}((\mathbb{R}^{\mathbf{n}} \setminus A_0) \cap \mathbf{B}^{\mathbf{n}}(\hat{y}, r)) \\ & \geq (2^{-1}\omega_{\mathbf{n}} - \epsilon)r^{\mathbf{n}} \end{aligned}$$

for r small enough depending on ϵ . This yields $B_0 \cap Q_0 = \emptyset$. Similarly we can show $B_0 \cap R_0 = \emptyset$, which proves (3.5).

Let $K \subset \mathbb{R}^{\mathbf{n}}$ be compact. Using (3.4), $R_0 \supset (\mathbb{R}^{\mathbf{n}} \setminus \bar{D})$ and the rectifiability of $\partial D = \bar{D} \setminus D$ we obtain

$$\mathcal{H}^{\mathbf{n}-1}(K \setminus (Q_0 \cup R_0)) \leq \mathcal{H}^{\mathbf{n}-1}((K \setminus D) \setminus R_0) \leq \mathcal{H}^{\mathbf{n}-1}(\partial D \cap K) < \infty.$$

In view of Federer [Fed69, §4.5.11] and [Fed69, §2.10.6] we can now use the general Gauss-Green theorem (see Federer [Fed69, §4.5.6]). Combined with (3.4) and (3.5) this establishes

$$\int_{A_0} \operatorname{div}_{\mathbb{R}^{\mathbf{n}}} \phi \, d\mathcal{L}^{\mathbf{n}} \leq \mathcal{H}^{\mathbf{n}-1}(D \cap B_0) \leq \mathcal{H}^{\mathbf{n}-1}(D \setminus (Q_0 \cup R_0)) = 0$$

for all $\phi \in \mathcal{C}_c^1(D, \mathbf{B}^{\mathbf{n}}(0, 1))$. Thus A_0 is stationary in D . Then the constancy theorem (see Allard [All72, Thm. 4.6.(3)] or Simon [Sim83, Thm. 8.4.1]) yields $A_0 = D$ which establishes the result. Hence it remains to prove (3.3).

We want to show $p[R_2 \cap \operatorname{graph} f] \subset R_0$. Consider $y \in R_2 \cap \operatorname{graph} f$. By (1.7) and as μ is integer \mathbf{n} -rectifiable we can estimate for $r \in (0, \infty)$

$$\begin{aligned} r^{-\mathbf{n}} \mathcal{L}^{\mathbf{n}}(A_0 \cap \mathbf{B}^{\mathbf{n}}(\hat{y}, r)) &= r^{-\mathbf{n}} \int_{\operatorname{graph} f \cap (A_0 \times \mathbb{R}^{\mathbf{k}}) \cap \mathbf{C}(y, r)} |JF \circ p|^{-1} \, d\mathcal{H}^{\mathbf{n}} \\ &\leq r^{-\mathbf{n}} \mu(\mathbf{B}(y, (1 + \operatorname{lip}(f))r)) \end{aligned}$$

and as $y \in R_2$ this goes to 0 for $r \searrow 0$. Thus $\hat{y} \in R_0$.

It remains to show $p[Q_2] \subset Q_0$. Suppose this is false, then there exists a $y_0 \in Q_2$, an $\epsilon \in (0, 1)$ and a sequence $(r_m)_{m \in \mathbb{N}}$ with $r_m \searrow 0$ such that

$$(3.6) \quad r_m^{-\mathbf{n}} \mathcal{L}^{\mathbf{n}}(\mathbf{B}^{\mathbf{n}}(\hat{y}_0, r_m) \setminus A_0) > 2\epsilon$$

for all $m \in \mathbb{N}$. Consider the sequence $(\mu_m)_{m \in \mathbb{N}}$ given by $\mu_m = \mu_{y_0, r_m}$. By (1.7), unit density and as $y_0 \in U_1$ we have

$$\begin{aligned} \limsup_{m \rightarrow \infty} \mu_m(\mathbf{B}(0, R)) &= \limsup_{m \rightarrow \infty} r_m^{-\mathbf{n}} \mu(\mathbf{B}(y_0, Rr_m)) \leq (1 + C \operatorname{lip} f)^{\mathbf{n}} R^{\mathbf{n}}, \\ \limsup_{m \rightarrow \infty} \|\delta \mu_m\|(\mathbf{B}(0, R)) &= \lim_{m \rightarrow \infty} r_m^{-\mathbf{n}-1} \|\delta \mu\|(\mathbf{B}(y_0, Rr_m)) = 0 \end{aligned}$$

for every $R \in (0, \infty)$. By varifold compactness (see Allard [All72, Thm. 6.4] or Simon [Sim83, Thm. 8.5.5]) there exists a stationary integer \mathbf{n} -rectifiable Radon measure ν with $0 \in \operatorname{spt} \nu$ and such that for a subsequence we have

$$(3.7) \quad \mathbf{V}(\mu_m) \rightarrow \mathbf{V}(\nu) \text{ as radon measures on } \mathbb{R}^{\mathbf{n}+\mathbf{k}} \times \mathbf{G}(\mathbf{n} + \mathbf{k}, \mathbf{n})$$

Define $f_m \in \mathcal{C}^{0,1}(\mathbf{B}^{\mathbf{n}}(0, 3), \mathbb{R}^{\mathbf{k}})$ by $f_m(\hat{x}) := r_m^{-1}(f(r_m \hat{x} + \hat{y}_0) - f(\hat{y}_0))$. By the Arzela-Ascoli theorem there exists a $g \in \mathcal{C}^{0,1}(\mathbf{B}^{\mathbf{n}}(0, 3), \mathbb{R}^{\mathbf{k}})$ such that for a subsequence $\|f_m - g\|_{C^0} \rightarrow 0$. We claim

$$(3.8) \quad \operatorname{spt} \nu \cap \mathbf{C}(0, 3) \subset \operatorname{graph}(g)$$

$$(3.9) \quad \operatorname{spt} \nu \cap \mathbf{C}(0, 1) = \operatorname{graph}(g) \cap \mathbf{C}(0, 1).$$

Suppose there exists a $z \in \text{spt}\nu \cap \mathbf{C}(0, 3) \setminus \text{graph}(g)$. Then we find $\rho \in (0, 1)$ with $\mathbf{B}(z, 4\rho) \cap \text{graph}(g) = \emptyset$ and $\nu(\mathbf{B}(z, \rho)) > 0$. Thus for some large enough $m \in \mathbb{N}$ we have $\mathbf{B}(z, 3\rho) \cap \text{graph}(f_m) = \emptyset$ and $\mu_m(\mathbf{B}(z, 2\rho)) > 0$. But by definition of f_m and μ_m combined with (1.7) we also have $\text{spt}\mu_m \subset \text{graph}(f_m)$, which yields a contradiction. Thus (3.8) holds.

Now suppose there exists a $z \in \text{graph}(g) \cap \mathbf{C}(0, 1) \setminus \text{spt}\nu$. Define

$$\rho_0 := \inf \{ \rho \in (0, \infty) : \mathbf{C}(z, \rho) \cap \text{spt}\nu \neq \emptyset \} > 0$$

Note that $\rho_0 < 1$ as $0 \in \text{spt}\nu$. Then there exists a $z_0 \in \partial\mathbf{C}(z, \rho_0) \cap \text{spt}\nu$ such that $\mathbf{C}(z, \rho_0) \cap \text{spt}\nu = \emptyset$. Consider the sequence $(\nu_l)_{l \in \mathbb{N}}$ given by $\nu_l = \nu_{z_0, r_l}$. As above there exists a stationary integer \mathbf{n} -rectifiable Radon measure η with $0 \in \text{spt}\eta$ and such that for a subsequence we have

$$(3.10) \quad \mathbf{V}(\nu_l) \rightharpoonup \mathbf{V}(\eta) \quad \text{as radon measures on } \mathbb{R}^{\mathbf{n}+\mathbf{k}} \times \mathbf{G}(\mathbf{n}+\mathbf{k}, \mathbf{n}).$$

Similar as above we also see

$$(3.11) \quad \text{spt}\eta \cap \mathbf{C}(0, 3) \subset \text{graph}(h) \quad \text{for some } h \in \mathcal{C}^{0,1}(\mathbf{B}^{\mathbf{n}}(0, 3), \mathbb{R}^{\mathbf{k}})$$

As ν is stationary and integer \mathbf{n} -rectifiable we know $\Theta^{\mathbf{n}}(\nu, z_0) \in [1, \infty)$ (see [All72, Cor. 4.3.3]). This yields that η satisfies (see [All72, Cor. 8.5.4])

$$\eta_{0,\lambda} = \eta \quad \text{for all } \lambda > 0,$$

in particular $x \in \mathbf{T}(\eta, x)$ for η -a. e. $x \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$. By choice of z_0 and (3.10) we also have $\text{spt}\eta \subset \{x \in \mathbb{R}^{\mathbf{n}+\mathbf{k}} : \hat{x} \cdot (\hat{z} - \hat{z}_0) \leq 0\}$, thus Lemma 3.2 yields $\text{spt}\eta \subset \{x \in \mathbb{R}^{\mathbf{n}+\mathbf{k}} : \hat{x} \cdot (\hat{z} - \hat{z}_0) = 0\}$. In view of (3.11) this implies $\mathcal{H}^{\mathbf{n}}(\text{spt}\eta) = 0$, hence $\text{spt}\eta = \emptyset$, which contradicts $0 \in \text{spt}\eta$. This proves (3.9).

We continue to lead (3.6) to a contradiction. Using (1.7) and the unit density of μ we can estimate

$$\begin{aligned} \mathcal{L}^{\mathbf{n}}(A_0 \cap \mathbf{B}^{\mathbf{n}}(\hat{y}_0, r_m)) &= \int_{\text{graph}(f) \cap (A_0 \times \mathbb{R}^{\mathbf{k}} \cap \mathbf{C}(y_0, r_m))} |JF \circ p|^{-1} d\mathcal{H}^{\mathbf{n}} \\ &\geq \int_{\mathbf{C}(y_0, r_m)} |\Lambda_{\mathbf{n}} \mathbf{T}(\mu, x)|^{-1} d\mu(x) = r_m^{\mathbf{n}} \int_{\mathbf{C}(y_0, 1)} |\Lambda_{\mathbf{n}} S_{\hat{\mathbf{t}}}|^{-1} d\mathbf{V}(\mu_m)(x, S). \end{aligned}$$

Recall ϵ from (3.6). In view of (3.7) and (3.9) we obtain

$$\begin{aligned} r_m^{-\mathbf{n}} \mathcal{L}^{\mathbf{n}}(A_0 \cap \mathbf{B}^{\mathbf{n}}(\hat{y}_0, r_m)) + \epsilon &\geq \int_{\mathbf{C}(0, 1)} |\Lambda_{\mathbf{n}} S_{\hat{\mathbf{t}}}|^{-1} d\mathbf{V}(\nu)(x, S) \\ &\geq \int_{\text{graph}(g) \cap \mathbf{C}(0, 1)} |JG \circ p|^{-1} d\mathcal{H}^{\mathbf{n}} = \omega_{\mathbf{n}} \end{aligned}$$

for m large enough, where $G(\hat{z}) := (\hat{z}, g(\hat{z}))$. Thus we see

$$r_m^{-\mathbf{n}} \mathcal{L}^{\mathbf{n}}(\mathbf{B}^{\mathbf{n}}(\hat{y}_0, r_m) \setminus A_0) = \omega_{\mathbf{n}} - r_m^{-\mathbf{n}} \mathcal{L}^{\mathbf{n}}(A_0 \cap \mathbf{B}^{\mathbf{n}}(\hat{y}_0, r_m)) \leq \epsilon,$$

which contradicts (3.6). This completes the proof of (3.3), which establishes the result. \square

3.2 Lemma. *Consider a stationary \mathbf{n} -rectifiable Radon measure η in $\mathbb{R}^{\mathbf{n}+\mathbf{k}}$ with $x \in \mathbf{T}(\eta, x)$ for η -a.e. $x \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$. Suppose $\text{spt}\eta \subset \{x \in \mathbb{R}^{\mathbf{n}+\mathbf{k}} : x \cdot w \leq 0\}$ for some $w \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$. Then $\text{spt}\eta \subset \{x \in \mathbb{R}^{\mathbf{n}+\mathbf{k}} : x \cdot w = 0\}$.*

Proof. Let $0 < R_1 < R_2 < \infty$ and consider $\psi \in \mathcal{C}^\infty(\mathbb{R}, [0, 1])$ non-increasing with $\psi(s) = 1$ on $(-\infty, R_1/2]$, $\psi(s) = 0$ on $[2R_2, \infty)$ and $\psi'(s) \leq (R_1 - R_2)/2$ on $[R_1, R_2]$. Define $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^{n+k}, [0, 1])$ by $\zeta(x) = \psi(|x|)$. As η is stationary and $x \in \mathbf{T}(\eta, x)$ for η -a.e. $x \in \mathbb{R}^{n+k}$ we can calculate

$$0 = \int_{\mathbb{R}^{n+k}} \operatorname{div}_\eta(\zeta w) \, d\eta = \int_{\mathbb{R}^{n+k}} \mathbf{T}(\eta, x)_\sharp(D\zeta) \cdot w \, d\eta = \int_{\mathbb{R}^{n+k}} \psi'(|x|) \frac{x \cdot w}{|x|} \, d\eta$$

By $\psi' \leq 0$ and our assumptions on $\operatorname{spt} \eta$ the integrand is always positive, moreover $\psi'(s) < 0$ on $[R_1, R_2]$, thus $x \cdot w = 0$ for η -a.e. $x \in \mathbf{B}(0, R_2) \setminus \mathbf{B}(0, R_1)$. This establishes the result. \square

4 Improved height bound

For a Brakke flow initial local height bounds in a certain direction yield weaker height bounds at later times in a decreased region. Such results can be obtained from sphere comparison, see Brakke [Bra78, Thm. 3.7] or Colding and Minicozzi [CM03, Lem. 3]. Here we prove stronger height bounds for short times using directly the Brakke flow equation.

4.1 Proposition. *For every $p \in \mathbb{N}$ there exists a $C_p \in (1, \infty)$ such that the following holds:*

Let $\rho \in (0, \infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1, \infty)$, $a \in \mathbb{R}^{n+k}$, $v \in \mathbb{R}^{n+k}$ and let $(\mu_t)_{t \in [t_1, t_2]}$ be a Brakke flow in $\mathbf{B}(a, 2\rho)$. Suppose

$$\operatorname{spt} \mu_{t_1} \cap \mathbf{B}(a, 2\rho) \subset \{x \in \mathbb{R}^{n+k} : (x - a) \cdot v \leq 0\}.$$

Then for all $t \in [t_1, t_2]$ we have

$$\operatorname{spt} \mu_t \cap \mathbf{B}(a, \rho) \subset \{x \in \mathbb{R}^{n+k} : (x - a) \cdot v \leq C_p |v| (t - t_1)^p \rho^{1-2p}\}.$$

Proof. We will actually prove the result for all $p \in \mathbb{N} \cup \{0\}$. For $p = 0$ the statement directly follows from $\mathbf{B}(a, \rho) \subset \{(x - a) \cdot v \leq \rho |v|\}$ with $C_0 = 1$. Suppose the statement is true for some $p \in \mathbb{N} \cup \{0\}$. We want to show the statement holds for $p + 1$. We may assume $\rho = 2$, $t_1 = 0$, $a = 0$ and $v = \mathbf{e}_{n+k}$. Set $N := n + k - 1$, $t_0 := \min\{2^{-7} n^{-1}, t_2\}$. In case $t_0 < t_2$ for $t \in (t_0, t_2]$ the conclusion directly follows from $\mathbf{B}(0, 2) \subset \{x \cdot \mathbf{e}_{n+k} \leq 2\}$ for C_{p+1} large enough. Hence we are only interested in $t \in [0, t_0]$. We claim

$$(4.1) \quad \operatorname{spt} \mu_t \cap \mathbf{B}(0, 2) \subset (\mathbf{B}^N(\bar{x}_0, 0) \times (-2, 1/4])$$

for all $t \in [0, t_0]$. In order to see this consider $z_0 = (\bar{x}, h) \in \mathbf{B}(0, 2)$ with $h > 1/4$. By assumption we have $\operatorname{spt} \mu_0 \cap \mathbf{B}(z_0, 1/4) = \emptyset$. Then Lemma 2.6 with $R = 1/8$ implies $z_0 \notin \operatorname{spt} \mu_t$ for all $t \in [0, t_0]$.

Fix an arbitrary $\bar{x}_0 \in \mathbf{B}^N(0, 2)$. By the induction assumption we see

$$(4.2) \quad \operatorname{spt} \mu_t \cap \mathbf{B}((\bar{x}_0, 0), 1) \subset (\mathbf{B}^N(\bar{x}_0, 0) \times (-1, C_p t^p))$$

for all $t \in [0, t_0]$. Consider the function $\eta \in \mathcal{C}^{0,1}(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}, \mathbb{R}^+) \cup \mathcal{C}^\infty(\{\eta > 0\})$ given by

$$\eta(t, \bar{x}, h) := (h - 8C_p |\bar{x} - \bar{x}_0|^2 t^p - C_{p+1} t^{p+1})_+.$$

Treat η as a function on $\mathbb{R} \times \mathbb{R}^{\mathbf{n}+\mathbf{k}}$. Using $\operatorname{div}_{\mu_t}((\mathbb{R}^N \times \{0\})_{\sharp}) \leq \mathbf{n}$ and choosing C_{p+1} large enough we can estimate

$$(4.3) \quad (\partial_t - \operatorname{div}_{\mu_t} D)\eta^3(t, x) \leq 3\eta^2(t, x)(16C_p \mathbf{n} t^p - (p+1)C_{p+1} t^p) \leq 0$$

for all $(t, x) \in ([0, t_2] \times \operatorname{spt} \mu_t) \cap \{\eta > 0\}$ at which the approximate tangent space exists.

Let $\chi \in \mathcal{C}^\infty(\mathbb{R}^{\mathbf{n}+\mathbf{k}}, [0, 1])$ be a cut-off function such that

$$\mathbf{B}((\bar{x}_0, 0), 1/2) \subset \{\chi = 1\} \subset \operatorname{spt} \chi \subset \mathbf{B}((\bar{x}_0, 0), 1).$$

In view of (4.1), (4.2) and by definition of η we see

$$(4.4) \quad \operatorname{spt} \mu_t \cap \operatorname{spt} \eta(t, \cdot) \cap \operatorname{spt} D\chi = \emptyset$$

for all $t \in [0, t_0]$. Consider the test function $\phi \in \mathcal{C}^2(\mathbb{R} \times \mathbb{R}^{\mathbf{n}+\mathbf{k}}, \mathbb{R}^+)$ given by $\phi := \eta^3 \chi$. Using (4.3), (4.4) and the Brakke flow equation (2.1) we obtain

$$\mu_s(\phi(s, \cdot)) - \mu_0(\phi(0, \cdot)) \leq \int_0^s \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \chi(\partial_t - \operatorname{div}_{\mu_t} D)\eta^3 \, d\mu_t \, dt \leq 0$$

for all $s \in (0, t_0]$. By assumption we have $\mu_0(\phi(0, \cdot)) = 0$, hence $\phi(t, x) = 0$ for all $t \in (0, t_0]$ and all $x \in \operatorname{spt} \mu_t$. By definition of η and χ this yields that $h \leq C_{p+1} t^p$ for all $t \in (0, t_0]$ and all $(\bar{x}, h) \in \operatorname{spt} \mu_t \cap (\{\bar{x}_0\} \times [0, 1/2])$. As $\bar{x}_0 \in \mathbf{B}^N(0, 2)$ was arbitrary we just proved

$$\operatorname{spt} \mu_t \cap (\mathbf{B}^N(0, 2) \times (-2, 1/2)) \subset (\mathbf{B}^N(0, 2) \times (-2, C_{p+1} t^{p+1}))$$

for all $t \in [0, t_0]$. In view of (4.1) this completes the statement for $p+1$ and the result follows by induction. \square

4.2 Corollary. *For every $p \in \mathbb{N}$ there exists a $c_p \in (0, 1)$ such that the following holds:*

Let $R_1, r_0 \in (0, \infty)$, $h_1 \in (0, r_0/8]$, $R_2 \in [r_0, \infty)$, $x_0 \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1, t_1 + c_p r_0^2)$ and let $(\mu_t)_{t \in [t_1, t_2]}$ be a Brakke flow in $\mathbf{C}(x_0, R_1 + r_0, R_2 + r_0)$. Suppose

$$\operatorname{spt} \mu_{t_1} \cap \mathbf{C}(x_0, R_1 + r_0, R_2 + r_0) \subset \mathbf{C}(x_0, R_1 + r_0, h_1).$$

Then for all $t \in [t_1, t_2]$ and $h(t) := h_1 + (t - t_1)^p r_0^{1-2p}$ we have

$$\operatorname{spt} \mu_t \cap \mathbf{C}(x_0, R_1, R_2) \subset \mathbf{C}(x_0, R_1, h(t)).$$

Proof. We may assume $t_1 = 0$ and $x_0 = 0$. Let $p \in \mathbb{N}$ be given and C_{p+1} be the value according to Proposition 4.1. First we want to show

$$(4.5) \quad \operatorname{spt} \mu_t \cap \mathbf{C}(0, R_1, r_0/4) \subset \mathbf{C}(0, R_1, h(t))$$

for all $t \in (0, t_2]$. To see this consider arbitrary $t \in (0, t_2]$ and $x \in \operatorname{spt} \mu_t \cap \mathbf{C}(0, R_1, r_0/4)$. We want to show $|\tilde{x}| \leq h(t)$. Suppose $\tilde{x} \neq 0$. Set $\tilde{v} := |\tilde{x}|^{-1} \tilde{x}$, $v := (0, \tilde{v}) \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$, and $a := (\hat{x}, h_1 \tilde{v})$. Note that $|x - a| = |\tilde{x}| - h_1 \leq r_0/4$.

For arbitrary $y \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ we have $(y - a) \cdot v \leq |\tilde{y}| - h_1$, hence by our initial height bound we see

$$\operatorname{spt} \mu_0 \cap \mathbf{B}(a, r_0/2) \subset \{y \in \mathbb{R}^{\mathbf{n}+\mathbf{k}} : (y - a) \cdot v \leq 0\}.$$

Using Proposition 4.1 with $\rho = r_0/4$ yields

$$|\tilde{x}| - h_1 = (x - a) \cdot v \leq (r_0/4)^{-2p-1} C_{p+1} t^{p+1} \leq t^p r_0^{1-2p},$$

as $t \leq t_2 \leq c_p r_0^2$ and for c_p small enough. As t and x were arbitrary, this establishes (4.5).

Now consider $z_0 \in \mathbf{C}(0, R_1, R_2)$ with $\tilde{z}_0 \geq r_0/4$. By our initial height bound and $h_1 \leq r_0/8$ we have $\text{spt}\mu_0 \cap \mathbf{B}(z_0, r_0/8) = \emptyset$. Thus Lemma 2.6 with $R = r_0/16$ implies $z_0 \notin \text{spt}\mu_t$ for all $t \in [0, t_2]$, where we used $t_2 \leq c_p r_0^2 \leq (2\mathbf{n})^{-1}(r_0/16)^2$. In view of (4.5), this completes the proof. \square

5 Maintain graphical representability

In this section we prove Theorem 1.3. The main idea of the proof is to iterate Brakkes local regularity theorem (see Theorem 2.10) by choosing a time at which graphical representation is obtained as the new starting time. To do so we first show a version of Theorem 2.10 which only has assumptions at the initial and final time, see Proposition 5.1.

By Corollary 4.2 initial height bounds yield weaker height bounds later on. Also by Huisken's monotonicity formula (see Theorem 2.8) initial bounds on area ratio imply bounds on area ratio in the future (see Lemma B.1). Moreover by the clearing out lemma (see Lemma 2.9) non-vanishing at some time yields a lower bound on measure a bit earlier. Thus with Brakke's local regularity theorem (see Theorem 2.10) we obtain the Proposition below, which is an improved version of a result found in the author's thesis [Lah14, Thm. 11.7].

5.1 Proposition. *For all $\kappa, \alpha \in (0, 1)$ there exists a $\sigma_1 \in (0, 1)$ such that the following holds:*

Let $\beta \in (0, 1]$, $\iota \in (0, \sigma_1\beta]$, $\varrho \in (0, \infty)$, $s_1 \in \mathbb{R}$, $s_2 \in (s_1 + 2\iota^2\varrho^2, s_1 + 4(\sigma_1\beta\iota)^\alpha\varrho^2]$, $z_0 \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and let $(\mu_t)_{t \in [s_1, s_2]}$ be a Brakke flow in $\mathbf{C}(z_0, 2\varrho, 2\varrho)$. Suppose

$$(5.1) \quad \text{spt}\mu_{s_2} \cap \mathbf{C}(z_0, \sigma_1\iota\varrho, \varrho) \neq \emptyset,$$

$$(5.2) \quad \text{spt}\mu_{s_1} \cap \mathbf{C}(z_0, 2\varrho, 2\varrho) \subset \mathbf{C}(z_0, 2\varrho, \sigma_1\beta\iota\varrho),$$

$$(5.3) \quad r^{-\mathbf{n}}\mu_{s_1}(\mathbf{B}(z_0, r)) \leq (2 - \kappa)\omega_{\mathbf{n}} \quad \text{for all } r \in (\sigma_1\iota\varrho, 8\mathbf{n}\sqrt{s_2 - s_1}].$$

Set $I := (s_1 + \iota^2\varrho^2, s_2 - \iota^2\varrho^2)$.

Then there exists an $u \in \mathcal{C}^\infty(I \times \mathbf{B}^{\mathbf{n}}(\hat{z}_0, \sigma_1\iota\varrho), \mathbb{R}^{\mathbf{k}})$ such that

$$\mu_t \llcorner \mathbf{C}(z_0, \sigma_1\iota\varrho, \varrho) = \mathcal{H}^{\mathbf{n}} \llcorner \text{graph}(u(t, \cdot)) \quad \text{for all } t \in I.$$

Moreover $\sup |Du| \leq \beta$ and $F_t(\hat{x}) = (\hat{x}, u(t, \hat{x}))$ satisfies (1.1).

Proof. We may assume $s_1 = 0$, $z_0 = 0$ and $\varrho = 1$. By Corollary 4.2 with $p = 2/\alpha$ and assumption (5.2) we have

$$(5.4) \quad \text{spt}\mu_t \cap \mathbf{C}(0, 1, 1) \subset \mathbf{C}(0, 1, 2\sigma_1\beta\iota)$$

for all $t \in [0, s_2]$, where we used $s_2^{2/\alpha} \leq 4^{2/\alpha}(\sigma_1\beta\iota)^2 \leq \sigma_1\beta\iota$ for σ_1 small enough.

Choose $\lambda_1 \in (0, \kappa]$ such that $\lambda_1 \leq (2C_1)^{-1}$ and $C_2(2^n \lambda_1 \omega_{\mathbf{n}})^{\frac{2}{n+6}} \leq (16\mathbf{n})^{-1}$, where C_1 and C_2 are the constants from Lemma B.1. and Lemma 2.9 respectively. Let $\Lambda \in (1, \infty)$ be from Theorem 2.10 chosen with respect to $\lambda = \lambda_1$. Consider the radius

$$\rho_1 := 4^{-1} \Lambda^{-\frac{1}{2}} \iota \in (2\sigma_1 \iota, \sqrt{s_2}),$$

where we chose σ_1 small enough. Set $t_2 := s_2 - (8\mathbf{n})^{-1}(\rho_1/2)^2$. We want to show

$$(5.5) \quad \rho_1^{-\mathbf{n}} \mu_{t_2}(\mathbf{B}(0, \rho_1/2)) \geq \lambda_1 \omega_{\mathbf{n}}.$$

Suppose this would be false, then we can use Lemma 2.9 with $\eta = 2^n \omega_{\mathbf{n}} \lambda_1$ and $R = \rho_1/2$ to obtain $\mu_{s_2}(\mathbf{B}(0, \rho_1/4)) = 0$. In view of (5.4) this contradicts (5.1). Thus (5.5) has to be true.

By assumption (5.3), Lemma B.1 and $\lambda_1 \leq \min\{\kappa, (2C_1 \omega_{\mathbf{n}})^{-1}\}$ we obtain

$$(5.6) \quad \rho_1^{-\mathbf{n}} \mu_t(\mathbf{B}(0, 4\rho_1)) \leq 2C_1 \omega_{\mathbf{n}} \leq \lambda_1^{-1},$$

$$(5.7) \quad \rho_1^{-\mathbf{n}} \mu_0(\mathbf{B}(0, \rho_1)) \leq (2 - \kappa) \omega_{\mathbf{n}} \leq (2 - \lambda_1) \omega_{\mathbf{n}}.$$

Now choose h_0 according to Theorem 2.10 with respect to $\lambda = \lambda_1$ as above. Set $h := \min\{h_0, \Lambda^{-1}\beta\}$. Note that $2\sigma_1 \beta \iota \leq h\rho_1$ for σ_1 small enough. Thus (5.4), (5.5), (5.6) and (5.7) let us apply Theorem 2.10 with $\rho = \rho_1$ which establishes the result. \square

Looking at Proposition 5.1 we see that at time $t_2 - 2\iota \varrho^2$ we satisfy a non-vanishing condition in an increased cylinder. This allows to iterate above Proposition to obtain graphical representability inside a larger cylinder.

5.2 Lemma. *For all $\kappa \in (0, 1)$ there exists a $\sigma_2 \in (0, 1)$ such that the following holds:*

Let $\beta \in (0, 1]$, $\lambda \in (0, \sigma_2 \beta]$, $s_4 \in (2\lambda \varrho_0^2, 2\sqrt[32]{\lambda} \varrho_0^2]$, $y_0 \in \mathbb{R}^{n+\mathbf{k}}$ and let $(\mu_t)_{t \in [0, s_4]}$ be a Brakke flow in $\mathbf{C}(y_0, 5\varrho_0, 2\varrho_0)$. Suppose we have

$$(5.8) \quad \text{spt} \mu_{s_4} \cap \mathbf{C}(y_0, \sigma_2 \lambda \varrho_0, \varrho_0) \neq \emptyset,$$

$$(5.9) \quad \text{spt} \mu_0 \cap \mathbf{C}(y_0, 5\varrho_0, 2\varrho_0) \subset \mathbf{C}(y_0, 5\varrho_0, \sigma_2 \beta \lambda \varrho_0),$$

$$(5.10) \quad r^{-\mathbf{n}} \mu_0(\mathbf{B}(y, r)) \leq (2 - \kappa) \omega_{\mathbf{n}}$$

for all $y \in \mathbf{B}^n(\hat{y}_0, 3\varrho_0) \times \{\tilde{y}_0\}$ and all $r \in (\sigma_2 \lambda \varrho_0, 8\mathbf{n} \sqrt{s_4}]$. Set $I := (\lambda \varrho_0^2, s_4 - \lambda \varrho_0^2)$.

Then there exists a $v \in C^\infty(I \times \mathbf{B}^n(\hat{y}_0, 2\varrho_0), \mathbb{R}^{\mathbf{k}})$ such that

$$\mu_t \llcorner \mathbf{C}(y_0, 2\varrho_0, \varrho_0) = \mathcal{H}^n \llcorner \text{graph}(v(t, \cdot)) \quad \text{for all } t \in I.$$

Moreover $\sup |Dv| \leq \beta$ and $F_t(\hat{x}) = (\hat{x}, v(t, \hat{x}))$ satisfies (1.1).

Proof. We may assume $y_0 = 0$, $\varrho_0 = 1$ and set $\alpha := 1/64$. Let σ_1 be from Proposition 5.1 with respect to κ . For $m \in \mathbb{N}$ set

$$R_m := m\sigma_1 \sqrt{\sigma_2} \lambda, \quad T_m := s_4 - 2m\sigma_2 \lambda^2, \quad J_m := (\lambda^2, T_m).$$

Note that $T_m = s_4 - 2\sigma_1^{-1} \sqrt{\sigma_2} R_m \lambda$, in particular for $R_m \leq 3$ and σ_2 small enough we have $T_m > s_4 - \lambda > 4\sigma_2 \lambda^2$.

Consider the following statement:
 $stat(m) :\Leftrightarrow$ There exists a $v_m \in \mathcal{C}^\infty(J_m \times \mathbf{B}^n(0, R_m), \mathbb{R}^k)$ such that

$$\begin{aligned} \mu_t \llcorner \mathbf{C}(0, R_m, 1) &= \mathcal{H}^n \llcorner \text{graph}(v_m(t, \cdot)) \quad \text{for all } t \in J_m, \\ \sup |Dv_m| &\leq \Lambda_2 \lambda \text{ and } F_t(\hat{x}) = (\hat{x}, v_m(t, \hat{x})) \text{ satisfies (1.1).} \end{aligned}$$

By Proposition 5.1 with $s_2 = s_4$, $\varrho = 1$, $\iota = \sqrt{\sigma_2} \lambda$ we see that $stat(1)$ is true. Now suppose $stat(m_0)$ holds for some $m_0 \in \mathbb{N}$ with $R_{m_0} \leq 3$. Using Proposition 5.1 with $s_2 = T_{m_0} - \sigma_2 \lambda^2$, $\varrho = 1$, $\iota = \sqrt{\sigma_2} \lambda$ and arbitrary $z_0 \in \mathbf{B}^n(0, R_{m_0} + \sigma_1 \sqrt{\sigma_2} \lambda) \times \{0\}^k$ yields that also $stat(m_0 + 1)$ is true. Thus $stat(m_1)$ holds for some $m_1 \in \mathbb{N}$ with $2 \leq R_{m_1} \leq 3$, which establishes the result. \square

Now consider a Brakke flow which is initially graphical with small Lipschitz constant. Then the conditions of Lemma 5.2 are satisfied for arbitrarily small scaling. Thus we can extend the interval of graphical representation up to the initial time.

5.3 Lemma. *There exists an $\sigma_3 \in (0, 1)$ such that the following holds:*

Let $\beta \in (0, 1]$, $\iota \in (0, \sigma_3 \beta]$, $\rho_0 \in (0, \infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1 + \iota \rho_0^2, t_1 + \sqrt[32]{\iota} \rho_0^2]$, $z_0 \in \mathbb{R}^{n+k}$ and let $(\mu_t)_{t \in [t_1, t_2]}$ be a Brakke flow in $\mathbf{C}(z_0, 5\rho_0, 2\rho_0)$. Assume $z_0 \in \text{spt} \mu_{t_1}$ and

$$(5.11) \quad \text{spt} \mu_{t_2} \cap \mathbf{C}(z_0, \sigma_3 \iota \rho_0, \rho_0) \neq \emptyset.$$

Suppose there exists an $u_0 \in \mathcal{C}^{0,1}(\mathbf{B}^n(\hat{z}_0, 5\rho_0), \mathbb{R}^k)$ with $\text{lip}(u_0) \leq \sigma_3 \beta \iota$ and

$$(5.12) \quad \mu_{t_1} \llcorner \mathbf{C}(z_0, 5\rho_0, 2\rho_0) = \mathcal{H}^n \llcorner \text{graph}(u_0).$$

Set $I := (t_1, t_2 - \iota \rho_0^2)$.

Then there exists an $u \in \mathcal{C}^\infty(I \times \mathbf{B}^n(\hat{z}_0, 2\rho_0), \mathbb{R}^k)$ such that

$$\mu_t \llcorner \mathbf{C}(z_0, 2\rho_0, \rho_0) = \mathcal{H}^n \llcorner \text{graph}(u(t, \cdot)) \quad \text{for all } t \in I.$$

Moreover $\sup |Du| \leq \beta$ and $F_t(\hat{x}) = (\hat{x}, u(t, \hat{x}))$ satisfies (1.1).

Proof. We may assume $t_1 = 0$, $z_0 = 0$ and $\rho_0 = 1$.

For $s \in (0, \iota/4]$ we consider the following statement:

$stat(s) :\Leftrightarrow$ There exists an $v_s \in \mathcal{C}^\infty((s, t_2 - \iota/4) \times \mathbf{B}^n(0, 1), \mathbb{R}^k)$ such that

$$(5.13) \quad \mu_t \llcorner \mathbf{C}(0, 2, 1) = \mathcal{H}^n \llcorner \text{graph}(v_s(t, \cdot)) \quad \text{for all } t \in (s, t_2 - \iota/4),$$

$$(5.14) \quad \sup |Dv_s| \leq \beta \text{ and } F_t(\hat{x}) = (\hat{x}, v_s(t, \hat{x})) \text{ satisfies (1.1)}$$

Suppose $stat(s_0)$ holds for some $s_0 \in (0, \iota/4]$. We want to show that in this case also $stat(\frac{s_0}{2})$ holds. Let $\hat{y} \in \mathbf{B}^n(0, 2)$ be arbitrary, set $y := (\hat{y}, u_0(\hat{y}))$ and $\varrho_0 := \sqrt{\iota^{-1} s_0} \leq 1/2$. Using assumption (5.12) and $\text{lip}(u_0) \leq \sigma_3 \beta \iota$ yields

$$(5.15) \quad \text{spt} \mu_0 \cap \mathbf{C}(y, 5\varrho_0, 3/2) \subset \mathbf{C}(y, 5\varrho_0, 5\sigma_3 \beta \iota \varrho_0).$$

Then by Corollary 4.2 with $R_1 = r_0 = 2\varrho_0$ and $R_2 = 5/4$ we obtain

$$(5.16) \quad \text{spt} \mu_t \cap \mathbf{C}((\hat{y}, 0), 2\varrho_0, 1) \subset \mathbf{C}(y, 2\varrho_0, \varrho_0)$$

for all $t \in [0, 2s_0]$. Here we estimated $5\sigma_3 \beta \iota \varrho_0 + \varrho_0^{-1} s_0 \leq C \iota \varrho_0 \leq \varrho_0$ and $|u_0(\hat{y})| \leq 1/4$.

Set $J_2 := (s_0/2, 3s_0/2)$. We want to use Lemma 5.2 with $\kappa = \frac{1}{2}$, $\lambda = \iota/2$, $s_4 = 2s_0$ and $y_0 = y$. Choosing σ_3 small enough we obtain the following: $\iota\varrho_0^2/2 \leq s_0/2$; Statement (5.15) implies (5.9); Using assumption (5.12) and $\text{lip}(u_0) \leq \sigma_3\beta\iota$, we see that (5.10) holds. Moreover by (5.16) and as $s_0 < 2s_0 < t_2 - \iota/4$ we can use assumption (5.13) to show (5.8). Then by Lemma 5.2 we obtain an $v_{s,\hat{y}} \in \mathcal{C}^\infty(J_2 \times \mathbf{B}^n(\hat{y}, 2\varrho_0), \mathbb{R}^k)$ with

$$\mu_t \llcorner \mathbf{C}((\hat{y}, 0), 2\varrho_0, 1) = \mu_t \llcorner \mathbf{C}(y, 2\varrho_0, \varrho_0) = \mathcal{H}^n \llcorner \text{graph}(v_{s,\hat{y}}(t, \cdot))$$

for all $t \in J_2$. Here we used (5.16) to obtain the first equality. Also $F(t, \hat{x}) = (\hat{x}, v_{s,\hat{y}}(t, \hat{x}))$ satisfies (1.1) and $\sup |Dv_{s,\hat{y}}| \leq \beta$. As $\hat{y} \in \mathbf{B}^n(0, 2)$ was arbitrary this shows $\text{stat}(\frac{s_0}{2})$ is true.

Similarly we can use Lemma 5.2 with $y_0 = 0$, $s_4 = t_2$, $\varrho_0 = 1$ and $\lambda = \iota/4$ to obtain that $\text{stat}(\iota/4)$ is true for σ_3 small enough. Hence we can start an iteration which yields that $\text{stat}(0)$ holds. This establishes the result. \square

Consider the situation of Lemma 5.3. If $\hat{z}_0 \times \mathbb{R}^k$ intersects μ_{t_2} and ι is small enough we have that $(\mu_t)_{t \in [t_2 - \iota\rho_0^2, t_2]}$ satisfies the conditions of Lemma 5.3 on the smaller scale $\rho_0/2$ with $\sigma_3\beta\iota$ replaced by β . Thus we can use $t_2 - \iota$ as the new starting time. This yields an iteration and by curvature bounds for graphical mean curvature flow, we can assure that the gradient does not blow up. This leads to the following:

5.4 Lemma. *There exists a $\sigma_4 \in (0, 1)$ such that the following holds:*

Let $\eta \in (0, \sigma_4]$, $\varrho_0 \in (0, \infty)$, $s_1 \in \mathbb{R}$, $s_2 \in (s_1, s_1 + \eta\varrho_0^2]$, $y_0 \in \mathbb{R}^{n+k}$ and let $(\mu_t)_{t \in [s_1, s_2]}$ be a Brakke flow in $\mathbf{C}(y_0, 2\varrho_0, 2\varrho_0)$. Assume $y_0 \in \text{spt}\mu_{s_1}$ and

$$(5.17) \quad \text{spt}\mu_{s_2} \cap (\{\hat{y}_0\}^n \times \mathbf{B}^k(\tilde{y}_0, \varrho_0)) \neq \emptyset.$$

Suppose there exists a $v_0 \in \mathcal{C}^{0,1}(\mathbf{B}^n(\hat{y}_0, 2\varrho_0), \mathbb{R}^k)$ with $\text{lip}(v_0) \leq \eta^4$ and

$$(5.18) \quad \mu_{s_1} \llcorner \mathbf{C}(y_0, 2\varrho_0, 2\varrho_0) = \mathcal{H}^n \llcorner \text{graph}(v_0).$$

Let $s \in (s_1, s_2)$ and $\varrho(s) := \eta^{-1/16} \sqrt{s_2 - s}$.

Then there exists a $v_s \in \mathcal{C}^\infty((s_1, s) \times \mathbf{B}^n(\hat{y}_0, \varrho(s)), \mathbb{R}^k)$ such that

$$\mu_t \llcorner \mathbf{C}(y_0, \varrho(s), \varrho_0) = \mathcal{H}^n \llcorner \text{graph}(v_s(t, \cdot)) \quad \text{for all } t \in (s_1, s).$$

Moreover $\sup |Dv_s| \leq \eta$ and $F_t(\hat{x}) = (\hat{x}, v_s(t, \hat{x}))$ satisfies (1.1).

Proof. We may assume $s_1 = 0$, $y_0 = 0$ and $\varrho_0 = 1$. For a smooth n -dimensional submanifold M and $\mu = \mathcal{H}^n \llcorner M$ set

$$\begin{aligned} \|\text{tilt}(\mu)\|_U &:= \sup_{x \in M \cap U} \|(\mathbb{R}^n \times \{0\}^k)_\sharp - \mathbf{T}(M, x)\| \\ \|\mathbf{A}(\mu)\|_U &:= \sup_{x \in M \cap U} |\mathbf{A}(M, x)| \end{aligned}$$

for $U \subset \mathbb{R}^{n+k}$ open. Here \mathbf{A} is the second fundamental form of M .

For $m \in \mathbb{N}$ set

$$\begin{aligned} \eta_0 &:= \sqrt[16]{\eta}, \quad \delta_0 := \eta_0^{-1} \sqrt{s_2}, \quad \delta_m := \eta_0^{2m+16} \delta_0, \\ \tau_0^+ &:= s_2 - 4\eta_0^{44} s_2, \quad \tau_m^+ := s_2 - \eta_0^{m+15} \delta_m^2, \\ \tau_0 &:= s_2 - 8\eta_0^{44} s_2, \quad \tau_m := s_2 - 2\eta_0^{m+15} \delta_m^2. \end{aligned}$$

Note that for all $m \in \mathbb{N} \cup \{0\}$

$$(5.19) \quad 0 < \tau_{m+1} - \tau_m^+ < \tau_{m+1}^+ - \tau_m \leq s_2 - \tau_m \leq C\eta_0^{m+10}\delta_{m+1}^2.$$

For $m \in \mathbb{N} \cup \{0\}$ we consider the following statement:
 $stat(m) : \Leftrightarrow$ There exists a $v_m \in \mathcal{C}^\infty((0, \tau_m^+) \times \mathbf{B}^n(0, 2\delta_m), \mathbb{R}^k)$ such that

$$(5.20) \quad \mu_t \llcorner U_m = \mathcal{H}^n \llcorner \text{graph}(v_m(t, \cdot)) \text{ for all } t \in (0, s)$$

$$(5.21) \quad F_t(\hat{x}) = (\hat{x}, v_m(t, \hat{x})) \text{ satisfies (1.1),}$$

$$(5.22) \quad \sup_{t \in (0, \tau_m^+)} \|\text{tilt}(\mu_t)\|_{U_m} \leq \sum_{i=0}^m 2^{-i-4}\eta,$$

$$(5.23) \quad \delta_{m+1}^2 \sup_{t \in [\tau_0/2, \tau_m]} \|\mathbf{A}(\mu_t)\|_{U_m}^2 \leq \eta_0^{2m+33},$$

where $U_m := \mathbf{C}(0, 2\delta_m, 1)$.

First we observe the following: Consider $t \in [0, s_2)$ then there exists an $i \in \mathbb{N} \cup \{0\}$ such that $\tau_i \geq t$ and

$$(5.24) \quad \eta_0^{-1} \sqrt{s_2 - t} \leq \delta_i.$$

If $t \in [0, \tau_0]$ this directly follows from the definition of δ_0 . For $t \in (\tau_0, s_2)$ choose $i \in \mathbb{N}$ such that (with (5.19))

$$s_2 - \tau_i \leq s_2 - t \leq s_2 - \tau_{i-1} \leq \eta_0^{i+8}\delta_i^2.$$

Also note that by (5.18), $\text{lip}(v_0) \leq \eta^4$, $\eta_0^{16} = \eta \leq \sigma_4$, $s_2 \leq \eta$ and Corollary 4.2 with $R_2 = 1$, $R_1 = r_0 = \eta_0^{-6}\delta_0 = \eta_0^{-7}\sqrt{s_2} \leq \eta_0$ and $p = 8$ we have

$$(5.25) \quad \text{spt}\mu_t \cap \mathbf{C}(0, \eta_0^{-6}\delta_0, 1) \subset \mathbf{C}(0, \eta_0^{-6}\delta_0, \eta^3\delta_0)$$

for all $t \in [0, s_2]$ where we chose σ_4 small enough.

Next notice that $stat(0)$ is true. To see this use Lemma 5.3 with $t_1 = 0$, $t_2 = s_2$, $\beta = \eta_0^{16}/16$, $\iota = \eta_0^{46}$ and $\rho_0 = 2\delta_0$. Use $\eta_0^{16} = \eta \leq \sigma_4$ and choose σ_4 small enough to obtain $\eta_0^{64} \leq \sigma_3\beta\iota$, $s_2 - \iota\rho_0^2 \geq \tau_0^+$ and $\sqrt[32]{\iota}\rho_0^2 \geq s_2$. In particular use (5.25) to see that $\text{spt}\mu_t \cap \mathbf{C}(0, 4\delta_0, 2\delta_0) = \text{spt}\mu_t \cap \mathbf{C}(0, 4\delta_0, 1)$. This shows (5.20)-(5.22) for $m = 0$ on the larger radius $4\delta_0$. By Lemma A.6 with $t_1 = 0$, $t_2 = \tau_0^+$ and $\rho = 2\delta_0$ we have

$$\sup_{t \in [\tau_0/2, \tau_0]} \|\mathbf{A}(\mu_t)\|_{\mathbf{C}(0, 2\delta_m, 1)} \leq C\tau_0^{-1/2} = Cs_2^{-1/2} = C/(\eta_0\delta_0) = C\eta_0^{17}/\delta_1,$$

which completes $stat(0)$.

Now consider $m \in \mathbb{N} \cup \{0\}$ such that $stat(m)$ holds, we want to show that then also $stat(m+1)$ holds. By (5.17) and (5.25) there exists a point a_ω in $\text{spt}\mu_{s_2} \cap (\{0\}^n \times \mathbf{B}^k(0, 1/4))$. Using (5.19) and Lemma 2.6 with $z_0 = a_\omega$ and $R = 4n\sqrt{s_2 - \tau_m}$ we see $\mathbf{B}(a_\omega, C\eta_0^5\delta_{m+1}) \cap \text{spt}\mu_{\tau_m} \neq \emptyset$. Now choose $a_m \in \text{spt}\mu_{\tau_m}$ as a nearest point to a_ω . In particular $a_m \in \mathbf{B}(a_\omega, C\eta_0^5\delta_{m+1}) \subset \mathbf{C}(0, C\eta_0^5\delta_{m+1}, 1/2)$ and $a_\omega - a_m \in \mathbf{T}(\mu_{\tau_m}, a_m)^\perp$. We claim

$$(5.26) \quad \text{spt}\mu_t \cap \mathbf{C}(0, \delta_m, 1) \subset \mathbf{C}(a_m, 2\delta_m, \delta_{m+1})$$

for all $t \in [\tau_m, s_2]$. For $m = 0$ this directly follows from (5.25), $|\hat{a}_0| \leq \delta_0$ and $\eta^3 \delta_0 = \eta_0^{30} \delta_1$. For $m \in \mathbb{N}$ note that by (5.20) and (5.22) we can use Corollary 4.2 with $t_1 = \tau_m$, $t_2 = s_2$, $R_1 = r_0 = \delta_m$, $R_2 = 1/4$ and $p = 2$ to see

$$\text{spt} \mu_t \cap \mathbf{C}(a_m, 2\delta_m, 1/4) \subset \mathbf{C}(a_m, 2\delta_m, C\eta\delta_m).$$

for all $t \in [\tau_m, s_2]$. Then (5.26) follows from (5.25), $|\hat{a}_m| \leq \delta_m$ and $C\eta\delta_m = C\eta_0^{14}\delta_{m+1} \leq \delta_{m+1}$ for σ_4 small enough.

Fix an $S \in \mathbf{SO}(\mathbf{n} + \mathbf{k})$ with $S(\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}}) = \mathbf{T}(\mu_{\tau_m}, a_m)$ and set

$$\nu_t := (S^{-1})_{\#}(\mu_t - a_m) \quad \text{for } t \in [0, s_2],$$

in particular $(\nu_t)_{t \in [0, s_2]}$ is a Brakke flow in $\mathbf{C}(0, 1, 1)$. Using (5.20) and (5.22) for σ_4 small enough there exists an $f \in \mathcal{C}^\infty((0, \tau_m^+) \times \mathbf{B}^{\mathbf{n}}(0, \delta_m), \mathbb{R}^{\mathbf{k}})$ with $\sup |Df| \leq \eta$ and

$$(5.27) \quad \nu_t \llcorner \mathbf{C}(0, \delta_m, \delta_m) = \mathcal{H}^{\mathbf{n}} \llcorner \text{graph}(f(t, \cdot)) \quad \text{for all } t \in (0, \tau_m^+).$$

By choice of a_ω and a_m we also have

$$(5.28) \quad S^{-1}(a_\omega - a_m) \in (\{0\}^{\mathbf{n}} \times \mathbf{B}^{\mathbf{k}}(0, C\eta_0^5 \delta_{m+1})) \cap \text{spt} \nu_{s_2}.$$

By definition of ν its evident that $f(\tau_m, 0) = 0$ and $Df(\tau_m, 0) = 0$. In view of (5.22) and (5.23) we can estimate

$$(5.29) \quad \sup_{\mathbf{B}^{\mathbf{n}}(0, 2^9 \delta_{m+1})} |Df(\tau_m, \cdot)| \leq C\eta_0^{m+33/2}.$$

Here we used Corollary [Lah15, Cor. A2], $\delta_{m+1} \leq \eta_0^2 \delta_m$, $\eta_0^{16} = \eta \leq \sigma_4$ and chose σ_4 small enough. In view of $0 \in \text{spt} \nu$, (5.19), (5.27), (5.29) we can use Corollary 4.2 to see

$$(5.30) \quad \text{spt} \nu_t \cap \mathbf{C}(0, 2^9 \delta_{m+1}, 2^9 \delta_{m+1}) \subset \mathbf{C}(0, 2^9 \delta_{m+1}, C\eta_0^{m+33/2} \delta_{m+1})$$

for all $t \in [\tau_m, s_2]$.

We want to use Lemma 5.3 with $t_1 = \tau_m$, $t_2 = s_2$, $\rho_0 = 2^6 \delta_{m+1}$, $\beta = l_1$, $\iota = 4^{-6} \eta_0^{m+16}$ and $z_0 = 0$, where l_1 is the from Lemma A.6. By $\eta_0^{16} = \eta \leq \sigma_4$ and for σ_4 small enough we see: (5.28) implies (5.11), (5.27) implies (5.12) and (5.29) yields the desired Lipschitz bound. Moreover with (5.19) we can estimate

$$\eta_0^{m+16} \delta_{m+1}^2 \leq s_2 - \tau_{m+1}^+ \leq s_2 - \tau_m \leq \sqrt[32]{\eta_0^{m+16}} \delta_{m+1}^2.$$

Then we obtain a $g \in \mathcal{C}^\infty((\tau_m, \tau_{m+1}^+) \times \mathbf{B}^{\mathbf{n}}(0, 2^7 \delta_{m+1}), \mathbb{R}^{\mathbf{k}})$ with

$$(5.31) \quad \nu_t \llcorner \mathbf{C}(0, 2^7 \delta_{m+1}, 2^7 \delta_{m+1}) = \mathcal{H}^{\mathbf{n}} \llcorner \text{graph}(g(t, \cdot))$$

for all $t \in (\tau_m, \tau_{m+1}^+)$ and $\sup |Dg| \leq l_1$. By (5.27), (5.31) and (5.19) we can use Lemma A.6 with $t_1 = \tau_{m+1}^+ - \delta_{m+1}^2 \geq \tau_0 - \delta_1 > 0$, $t_2 = \tau_{m+1}^+$ and $\rho = 2^6 \delta_{m+1}$ to obtain

$$(5.32) \quad \sup_{t \in [\tau_m, \tau_{m+1}^+]} \|\mathbf{A}(\nu_t)\|_{\mathbf{C}(0, 2^6 \delta_{m+1}, 2^7 \delta_{m+1})}^2 \leq C\delta_{m+1}^{-2}.$$

Here we used (5.19) to estimate

$$\tau_m - (\tau_{m+1}^+ - \delta_{m+1}^2) \geq -C\eta_0^{m+10}\delta_{m+1}^2 + \delta_{m+1}^2 \geq \delta_{m+1}^2/2.$$

By definition of ν we have $\|\mathbf{A}(\nu_t)\|_{\mathbf{B}(0,2^6\delta_{m+1})}^2 \leq \|\mathbf{A}(\mu_t)\|_{\mathbf{C}(0,\delta_m,1)}^2$ for all $t \in (0, \tau_m]$. Then in view of (5.19), (5.23), (5.27), (5.31) and (5.32) we can use Lemma A.5 with $t_1 = \tau_m$, $t_2 = \tau_{m+1}^+$, $\varrho = 2^5\delta_{m+1}$, $L = C$ and $p = 4$ to estimate

$$(5.33) \quad \delta_{m+1}^2 \|\mathbf{A}(\nu_t)\|_{\mathbf{B}(0,2^5\delta_{m+1})}^2 \leq C(\eta_0^{2m+33} + \eta_0^{4(m+10)}) \leq C\eta_0^{2m+33}$$

for all $t \in [\tau_m, \tau_{m+1}^+)$.

In view of (5.30), (5.31) and (5.33) we can use Lemma [Lah15, Lem. A4] to obtain the gradient bound

$$(5.34) \quad \sup_{[\tau_m, \tau_{m+1}^+) \times \mathbf{B}^n(0,2^5\delta_{m+1})} |Dg| \leq C\eta_0^{m+33/2}.$$

By definition of ν , (5.22), (5.31) and (5.34) there exists a smooth function $h \in \mathcal{C}^\infty((\tau_m, \tau_{m+1}^+) \times \mathbf{B}^n(0, 8\delta_{m+1}), \mathbb{R}^k)$ with

$$\begin{aligned} \mu_t \llcorner \mathbf{C}(a_m, 8\delta_{m+1}, \delta_{m+1}) &= \mathcal{H}^n \llcorner \text{graph}(h(t, \cdot)) \\ \|\text{tilt}(\mu_t)\|_{\mathbf{C}(a_m, 8\delta_{m+1}, \delta_{m+1})} &\leq \sum_{i=0}^m 2^{-i-4}\eta + C\eta_0^{m+33/2} \leq \sum_{i=0}^{m+1} 2^{-i-4}\eta, \end{aligned}$$

for all $t \in (\tau_m, \tau_{m+1}^+)$, where we used $\eta_0^{16} = \eta \leq \sigma_4$ and chose σ_4 small enough. By (5.26) and $|a_m| \leq \delta_{m+1}$ this proves (5.20)-(5.22) for $m+1$. Also

$$\|\mathbf{A}(\mu_t)\|_{\mathbf{C}(0,4\delta_{m+1},1)}^2 \leq \|\mathbf{A}(\mu_t)\|_{\mathbf{B}(a_m,2^5\delta_{m+1})}^2 = \|\mathbf{A}(\nu_t)\|_{\mathbf{B}(0,2^5\delta_{m+1})}^2$$

for all $t \in [\tau_m, \tau_{m+1}]$. Hence (5.33) and $\delta_{m+1} = \eta_0^{-2}\delta_{m+2}$ imply (5.23) for $m+1$. This proves *stat*($m+1$) and in view of (5.24) an induction establishes the result. \square

Consider the setting of Theorem 1.3. We will use Lemma 5.4 to show that for all $t \in (t_1, t_2)$ we have that $\text{spt}\mu_t \cap \mathbf{C}(a, \rho, \rho)$ is contained in a Lipschitz graph and has density < 2 almost everywhere. Then by Theorem 1.5 and the properties of a Brakke flow we find a sequence $\tau_m \nearrow t_2$ such that $\text{spt}\mu_{\tau_m} \llcorner \mathbf{C}(a, \rho, \rho)$ is graphical. This allows us to apply Lemma 5.4 with arbitrary centre point in $\mathbf{B}^n(\hat{a}, \rho) \times \{0\}^k$ and final time τ_m to conclude $\text{spt}\mu_t \llcorner \mathbf{C}(a, \rho, \rho)$ is graphical for all $t \in [t_1, \tau_m)$. By the convergence of (τ_m) this establishes Theorem 1.3.

Proof of Theorem 1.3. We may assume $a = 0$, $t_1 = 0$ and $\rho = 2$. Using (1.6) and Corollary 4.2 we obtain

$$(5.35) \quad \text{spt}\mu_t \cap \mathbf{C}(0, 3, 3) \subset \mathbf{C}(0, 3, 7l_0)$$

for all $t \in [0, t_2]$.

Set $U := \mathbf{B}^n(0, 2) \times \mathbb{R}^k$. By definition of a Brakke flow we find a sequence $(\tau_m)_{m \in \mathbb{N}}$ with $\tau_m \nearrow t_2$, $\tau_m \in (0, t_2]$ such that for all $m \in \mathbb{N}$ we have $\mu_m := \mu_{\tau_m} \llcorner \mathbf{C}(0, 2, 2)$ is integer \mathbf{n} -rectifiable and the generalised mean curvature vector

\mathbf{H}_{μ_m} inside U exists. In particular $\|\partial\mu_m\|$ is absolutely continuous with respect to μ_m . Fix an arbitrary $m \in \mathbb{N}$. We want to show

$$(5.36) \quad \text{spt}\mu_m \cap U \subset \text{graph}(f_m)$$

for some Lipschitz function $f_m : \mathbf{B}^n(0, 4) \rightarrow \mathbb{R}^k$.

Let $x, y \in \text{spt}\mu_m \cap U$ with $x \neq y$. Set $y_0 := (\hat{y}, u_0(\hat{y}))$. We want to show $|\tilde{x} - \tilde{y}| \leq L|\hat{x} - \hat{y}|$ for some constant $L \in (1, \infty)$ which will depend on l_0 . By (5.35) we have $|\tilde{x} - \tilde{y}| \leq 7l_0$. Hence we may assume $|\hat{x} - \hat{y}| \leq l_0$.

First consider the case $\tau_m \leq 4|\hat{x} - \hat{y}|^2 \leq 4l_0^2$ and let $z \in \text{spt}\mu_m \cap U$. Then $\mu_m(\mathbf{B}(z, 2\sqrt{n\tau_m})) > 0$, so by Lemma 2.6 we have $\mu_0(\mathbf{B}(z, 4\sqrt{n\tau_m})) > 0$. Thus by (1.6) and $\text{lip}(u_0) \leq l_0$ we have $|\tilde{z} - \tilde{y}_0| \leq l_0|\hat{z} - \hat{y}_0| + 8\sqrt{n\tau_m}$. For $z = x, y$ this yields the wanted estimate.

Now consider the case $0 < 4|\hat{x} - \hat{y}|^2 < \tau_m$. Set $\epsilon := |\hat{x} - \hat{y}|$. By (5.35) we have $y \in \text{spt}\mu_m \cap (\{\hat{y}_0\} \times \mathbf{B}^k(0, 1))$. Set $s_m := \tau_m - 2\epsilon^2$. Using Lemma 5.4 with $s_1 = 0$, $s_2 = \tau_m$, $\varrho_0 = 1$, $\eta = \sqrt[4]{l_0}$ we obtain a $v_m \in \mathcal{C}^\infty(\mathbf{B}^n(\hat{y}, 8n\epsilon))$ with $\sup|Dv_m| \leq \sqrt[4]{l_0} \leq 1$ and

$$(5.37) \quad \text{spt}\mu_{s_m} \cap \mathbf{C}(y_0, 8n\epsilon, 1) = \mathcal{H}^n \llcorner \text{graph}(v_m).$$

Consider $z \in \text{spt}\mu_m \cap U$ with $|\hat{z} - \hat{y}| \leq \epsilon$. Then $\mu_m(\mathbf{B}(z, 2\sqrt{n\epsilon})) > 0$, so by Lemma 2.6 we have $\mu_{s_m}(\mathbf{B}(z, 4\sqrt{n\epsilon})) > 0$. In view of (5.35) we can use (5.37) to estimate $|\tilde{z} - v_m(\hat{y})| \leq (1 + 8n)\epsilon$. For $z = x, y$ this proves (5.36).

Next we want to show that μ_m has unit density. Let $y \in \text{spt}\mu_m \cap U$ and $r \in (0, \sqrt{\tau_m})$ be given. Set $s_r := \tau_m - 16\sqrt[32]{l_0}r^2$ and $y_0 := (\hat{y}, u_0(\hat{y}))$. Note that by (5.35) we have $y \in \text{spt}\mu_m \cap (\{\hat{y}_0\} \times \mathbf{B}^k(0, 1))$. Using Lemma 5.4 with $s_1 = 0$, $s_2 = \tau_m$, $\varrho_0 = 1$, $\eta = \sqrt[4]{l_0}$ we obtain a $v_r \in \mathcal{C}^\infty(\mathbf{B}^n(\hat{y}, 4r))$ with $\sup|Dv_r| \leq \sqrt[4]{l_0}$ and

$$(5.38) \quad \text{spt}\mu_{s_r} \cap \mathbf{C}(y_0, 4r, 1) = \mathcal{H}^n \llcorner \text{graph}(v_r),$$

Consider a radial cut-off function $\zeta_r \in \mathcal{C}_c^\infty(\mathbb{R}^{n+k}, [0, 1])$ with $\sup|D^2\zeta_r| \leq Cr^{-2}$ and

$$\zeta_r(x) = \begin{cases} 1 & \text{for } 0 \leq |x - y| \leq r \\ 0 & \text{for } (1 + 2^{-n-2})r \leq |x - y|. \end{cases}$$

Using equation (2.1), Remark 2.2 and Lemma 2.6 we estimate

$$\begin{aligned} & \mu_m(\mathbf{B}(y, r)) - \mu_{s_r}(\mathbf{B}(y, (1 + 2^{-n-2})r)) \\ & \leq \mu_m(\zeta_r) - \mu_{s_r}(\zeta_r) \leq C \int_{s_r}^{\tau_m} (\sup|D^2\zeta_r| \mu_t(\{\zeta > 0\})) \, dt \\ & \leq C \sqrt[32]{l_0} \sup\{\mu_t(\mathbf{B}(y, 2r)), t \in [s_r, \tau_m]\} \leq C \sqrt[32]{l_0} \mu_{s_r}(\mathbf{B}(y, 4r)). \end{aligned}$$

In view of (5.35) we have $|\tilde{y} - \tilde{y}_0| \leq 1/4 < 1 - 4r$, hence we can use (5.38) and the above estimate to obtain

$$\mu_m(\mathbf{B}(y, r)) \leq (1 + C\sqrt[4]{l_0}) \left(C \sqrt[32]{l_0} (4r)^n + ((1 + 2^{-n-2})r)^n \right) \leq \frac{3}{2} r^n,$$

where we chose l_0 small enough. As we already know μ_m is integer rectifiable, this shows that μ_m even has unit density in U . Also, by (1.5) and Lemma 2.6

we have $\text{spt}\mu_m \cap U \neq \emptyset$. Then Theorem 1.5 yields that in (5.36) actually holds equality. Hence

$$\mu_{\tau_m} \llcorner \mathbf{C}(0, 2, 2) = \mathcal{H}^n \llcorner \text{graph} f_m$$

for all $m \in \mathbb{N}$, for some Lipschitz function $f_m : \mathbf{B}^n(0, 2) \rightarrow \mathbb{R}^k$. In view of this and (5.35) we can use Lemma 5.4 with $s_2 = \tau_m$, $\varrho_0 = 1$ and arbitrary $y_0 \in \mathbf{B}^n(0, 2) \times \{0\}^k$ to obtain graphical representability inside $\mathbf{C}(0, 2, 2)$ for times in $(0, \tau_m)$. As $\tau_m \nearrow t_2$ this actually holds on $(0, t_2)$. Finally for the Lipschitz bound consider $(t, \hat{y}) \in (0, t_2) \times \mathbf{B}^n(0, 2)$. Lemma 5.4 with $y_0 := (\hat{y}, u_0(\hat{y}))$, $\rho_0 = 1$, $s_2 = t + \epsilon$ and $\eta = \sqrt[4]{l} + t + \epsilon$ yields $|Du(t, \hat{y})| \leq \sqrt[4]{l} + t + \epsilon$ for all $\epsilon \in (0, t_2 - t)$. Letting $\epsilon \searrow 0$ completes the result. \square

6 Brakke-type local regularity

Here we proof Theorem 1.1. First note that under slightly stronger assumptions on the starting density ratios the result directly follows from Lemma 5.2 and Theorem 1.3, see below:

6.1 Lemma. *There exists a constant $\sigma_5 \in (0, 1)$ and for every $\kappa \in (0, 1)$ exists an $h_2 \in (0, \sigma_5^2)$ such that the following holds:*

Let $h \in (0, h_2]$, $\varrho \in (0, \infty)$, $s_1 \in \mathbb{R}$, $s_2 \in (s_1 + \sqrt{h}\varrho^2, s_1 + \sigma_5\varrho^2]$, $x_0 \in \mathbb{R}^{n+k}$ and let $(\mu_t)_{t \in [s_1, s_2]}$ be a Brakke flow in $\mathbf{C}(x_0, 2^5\varrho, 2^5\varrho)$. Suppose $x_0 \in \text{spt}\mu_{s_2}$,

$$(6.1) \quad \text{spt}\mu_{s_1} \cap \mathbf{C}(x_0, 2^5\varrho, 2^5\varrho) \subset \mathbf{C}(x_0, 2^5\varrho, h\varrho),$$

$$(6.2) \quad r^{-n}\mu_{s_1}(\mathbf{B}(y, r)) \leq (2 - \kappa)\omega_n$$

for all $y \in \mathbf{B}^n(\hat{x}_0, 2^4\varrho) \times \{\hat{x}_0\}$ and all $r \in (h\varrho, \varrho)$. Set $I := (s_1 + \sqrt{h}\varrho^2, s_2)$.

Then there exists a $v \in C^\infty(I \times \mathbf{B}^n(\hat{x}_0, \varrho), \mathbb{R}^k)$ such that

$$\mu_t \llcorner \mathbf{C}(x_0, \varrho, \varrho) = \mathcal{H}^n \llcorner \text{graph}(v(t, \cdot)) \quad \text{for all } t \in I.$$

Moreover $\sup |Dv(t, \cdot)| \leq \sqrt[16]{h} + \varrho^{-2}(t - s_1)$ for all $t \in I$ and $F_t(\hat{x}) = (\hat{x}, v(t, \hat{x}))$ satisfies (1.1).

Proof. We may assume $x_0 = 0$, $s_1 = 0$ and $\varrho = 1$. First note that by assumption (6.1), $s_2 \leq \sigma_5$, $h \leq h_2 < \sigma_5$ and Corollary 4.2 with $R_1 = R_2 = r_0 = 16$ we have

$$(6.3) \quad \text{spt}\mu_t \cap \mathbf{C}(0, 16, 16) \subset \mathbf{C}(0, 16, 2\sigma_5),$$

for all $t \in [0, s_2]$.

Let σ_2 be from Lemma 5.2 with respect to κ and set $s_4 := \sqrt{h} < s_2$, $J := (\sqrt{h}/4, s_4 - \sqrt{h}/4)$. Lemma 2.6 and $0 \in \text{spt}\mu_{s_2}$ yield the existence of a $z_0 \in \text{spt}\mu_{s_4} \cap \mathbf{B}(0, 1)$. By Lemma 5.2 with $\varrho_0 = 4$, $y_0 = (\hat{z}_0, 0)$, $\beta = 2^4\sigma_2^{-1}\sqrt{h}$ and $\lambda = 2^{-6}\sqrt{h}$ there exists a $v_1 \in C^\infty(J \times \mathbf{B}^n(\hat{z}_0, 8), \mathbb{R}^k)$ such that

$$\mu_t \llcorner \mathbf{C}((\hat{z}_0, 0), 8, 4) = \mathcal{H}^n \llcorner \text{graph}(v_1(t, \cdot)) \quad \text{for all } t \in J.$$

Moreover $\sup |Dv_1| \leq 2^4\sigma_2^{-1}\sqrt{h}$. Here we chose $h_2 \leq \sigma_2^{-2}$. Then (6.3), $|\hat{z}_0| \leq 1$ and Theorem 1.3 with $t_1 = \sqrt{h}/2$, $a = (0, v_1(t_1, 0))$, $l = 2^4\sigma_2^{-1}\sqrt{h}$ and $\rho = 2$ yield the result. Here we chose h_2 small depending on σ_2 and l_0 . \square

Now under the assumptions of Theorem 1.1 we can find a time s_1 shortly after t_1 such that $\mu_{s_1} \ll \mathbf{C}(a, \rho, \rho)$ has bounded mean-curvature-excess and still has small height. By Lemma 2.12 then also the tilt-excess has to be small. Thus Brakke's cylindrical growth theorem (see Theorem 2.13) can be used to show that the density assumptions of Lemma 6.1 hold, which then yields the conclusion of Theorem 1.1.

proof of Theorem 1.1. We may assume $a = 0$, $t_1 = 0$ and $\rho = 1$. First consider the case $\gamma > 0$. Set $U := \mathbf{B}(0, 1)$ and $C(x, r) := \mathbf{C}(x, r) \cap U$ for $r \in (0, \infty)$, $x \in U$. In view of assumption (1.2) and as $\mathbf{C}(0, \sqrt{2}, \sqrt{2}) \subset \mathbf{B}(0, 2)$ we can use Corollary 4.2 with $r_0 = \sqrt{2} - 1$ and $p = 4$ to obtain

$$(6.4) \quad \text{spt} \mu_t \cap \mathbf{C}(0, 1, 1) \subset \mathbf{C}(0, 1, 2\gamma)$$

for all $t \in [0, \sqrt[4]{\gamma}]$ for γ_0 small enough. Fix a $\sigma \in (0, 2^{-5})$ such that $(1 - 8\sigma)^{-n} \leq 1 + \lambda/8$ and $(1 + 4\sigma)^n \leq 1 + \lambda/32$. In particular in the following γ_0 may depend on σ . By Lemma 2.6 and assumption (1.3) we can estimate

$$(6.5) \quad \mu_t(\mathbf{B}(0, 1 - \sigma)) \leq C\sigma^{-n-k} \mu_0(\mathbf{B}(0, 1)) \leq C\sigma^{-n-k}$$

for all $t \in [0, \sqrt[4]{\gamma}]$.

Fix a cut-off function $\psi \in C^\infty(\mathbb{R}, [0, 1])$ with $|\psi''| \leq C\sigma^{-2}$ and

$$\psi(t) = \begin{cases} 1 & \text{for } 0 \leq |t| \leq 1 - 2\sigma \\ 0 & \text{for } 1 - \sigma \leq |t|. \end{cases}$$

Consider $\zeta \in C_c^\infty(\mathbf{B}(0, 1), [0, 1])$ given by $\zeta(x) = \psi(|x|)$. For $s \in (0, \sqrt[4]{\gamma}]$ equation (2.1) and Remark 2.2 yield

$$\begin{aligned} D &:= \mu_s(\zeta) + \frac{1}{2} \int_0^s \int_{\mathbb{R}^{n+k}} |\mathbf{H}_{\mu_t}|^2 \zeta \, d\mu_t \, dt \\ &\leq \mu_0(\zeta) + \sup |D^2 \zeta| \int_0^s \mu_t(\{\zeta > 0\}) \, dt. \end{aligned}$$

Hence by (1.3) and (6.5) we have

$$D \leq (2 - \lambda)\omega_n + Cs\sigma^{-n-k-2} \leq (2 - \lambda/2)\omega_n,$$

where we used $s \leq \sqrt[4]{\gamma_0}$ and we chose γ_0 small enough. By (6.4) we have $\{\zeta = 1\} \supset \mathbf{B}(0, 1 - 2\sigma) \supset \text{spt} \mu_s \cap C(0, 1 - 4\sigma)$, for $\gamma_0 \leq \sigma$. Thus

$$(6.6) \quad \mu_s(C(0, 1 - 4\sigma)) + \frac{1}{2} \int_0^s \int_{C(0, 1 - 4\sigma)} |\mathbf{H}_{\mu_t}|^2 \, d\mu_t \, dt \leq D \leq (2 - \lambda/2)\omega_n$$

for all $s \in (0, \sqrt[4]{\gamma}]$. In particular we find an $s_1 \in (0, \sqrt[4]{\gamma}]$ such that $\mu_{s_1} \ll U$ is integer n -rectifiable, has \mathcal{L}^2 -integrable mean curvature vector and

$$(6.7) \quad \int_{C(0, 1 - 4\sigma)} |\mathbf{H}_{\mu_{s_1}}|^2 \, d\mu_{s_1} \leq 2(2 - \lambda/2)\omega_n \gamma^{-1/4} \leq C\gamma^{-1/4}.$$

Consider $y \in \mathbf{B}(0, \sigma)$. By (6.6) and choice of σ we can estimate

$$(6.8) \quad \mu_{s_1}(C(y, 1 - 8\sigma)) \leq (2 - \lambda/2)\omega_n \leq (2 - \lambda/4)\omega_n(1 - 8\sigma)^n.$$

Let $f \in \mathcal{C}_c^\infty(C(0, 1 - 4\sigma), [0, 1])$ be such that $f(x) = \psi((1 - 4\sigma)^{-1}|\hat{x}|)$ for $x \in \text{spt}\mu_{s_1} \cap U$. In view of (6.4), (6.6) and (6.7) we can use Lemma 2.12 with $f = g = h$ to obtain

$$(6.9) \quad \int_{C(0, 1-6\sigma)} \|(\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}})_{\mathfrak{h}} - \mathbf{T}(\mu_{s_1}, x)_{\mathfrak{h}}\|^2 d\mu_{s_1}(x) \leq C\gamma^{7/8}.$$

Here we estimated $\sup |Df|^2 \leq C\sigma^{-2} \leq C\gamma_0^{-1} \leq C\gamma^{-1}$.

Consider $y \in \mathbf{B}(0, \sigma)$ and $r_0 \in (2^{-5}\gamma^{16\alpha_0}\sigma, \sigma)$. Let $R_2 = 1 - 8\sigma$ and $R_1 = (1 + 4\sigma)r_0$. By (6.7) and (6.9) the assumptions of Theorem 2.13 are satisfied for $\alpha^2 = C\gamma^{-1/4}(\gamma^{16\alpha_0}\sigma)^{-\mathbf{n}}$ and $\beta^2 = C\gamma^{7/8}(\gamma^{16\alpha_0}\sigma)^{-\mathbf{n}}$. Hence we can estimate

$$\begin{aligned} & \left| R_2^{-\mathbf{n}} \int_U \psi(R_2^{-1}|\hat{x} - \hat{y}|) d\mu_{s_1}(x) - R_1^{-\mathbf{n}} \int_U \psi(R_1^{-1}|\hat{x} - \hat{y}|) d\mu_{s_1}(x) \right| \\ & \leq C\gamma^{-1/8}(\gamma^{16\alpha_0}\sigma)^{-\mathbf{n}}\gamma^{7/16} \leq C\sigma^{-\mathbf{n}}\gamma_0^{3/16} \leq \lambda\omega_{\mathbf{n}}/8, \end{aligned}$$

where we chose α_0 and γ_0 small enough. By estimate (6.8), definition of ψ and σ this yields

$$((1 + 4\sigma)r_0)^{-\mathbf{n}}\mu_{s_1}(\mathbf{B}(y, r_0)) \leq (2 - \lambda/8)\omega_{\mathbf{n}} \leq (2 - \lambda/16)\omega_{\mathbf{n}}(1 + 4\sigma)^{-\mathbf{n}}.$$

Now we can use Lemma 6.1 with $s_2 = t_2$, $\kappa = \lambda/16$, $\varrho = 2^{-5}\sigma$ and $h = \gamma^{16\alpha_0}$ to obtain the result. For the case $\gamma = 0$ use the above result with arbitrary small γ . \square

7 White-type local regularity

Here we want to prove Theorem 1.2. First we observe that a Brakke flow for which all Gaussian density ratios are one, has to be a plane. This mainly follows from Huisken's monotonicity formula (see Theorem 2.8).

7.1 Lemma. *Let $M \in (1, \infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1, \infty)$ and $(\mu_t)_{t \in [t_1, t_2]}$ be a Brakke flow in $\mathbb{R}^{\mathbf{n}+\mathbf{k}}$. Suppose $\text{spt}\mu_{t_2} \neq \emptyset$ and for all $(s, y) \in (t_1, t_2] \times \mathbb{R}^{\mathbf{n}+\mathbf{k}}$*

$$(7.1) \quad \sup_{R \in (0, \infty)} R^{-\mathbf{n}}\mu_s(\mathbf{B}(y, R)) \leq M,$$

$$(7.2) \quad \sup_{t \in [t_1, s]} \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(s, y)} d\mu_t \leq 1.$$

Then there exists a $T \in \mathbf{G}(\mathbf{n} + \mathbf{k}, \mathbf{n})$ and an $a \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ such that $\mu_t = \mathcal{H}^{\mathbf{n}} \llcorner (T + a)$ for all $t \in (t_1, t_2)$.

Proof. We may assume $t_1 = -1$ and $t_2 = 0$. For $t \in (-1, 0)$ let $D(t)$ be the set of all $y \in \text{spt}\mu_t$ such that $\Theta^{\mathbf{n}}(\mu_t, y) \geq 1$ and $\mathbf{T}(\mu_t, y)$ exists. Fix $s \in (-1, 0)$ and $y \in D(s)$. For $\epsilon \in (0, 1)$ there exist a radial symmetric cut-off function $\zeta \in \mathcal{C}_c^0(\mathbb{R}^{\mathbf{n}+\mathbf{k}}, [0, 1])$ such that

$$(7.3) \quad \int_{\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}}} \Phi_{(0,0)}(-1, x)\zeta(x) d\mathcal{H}^{\mathbf{n}}(x) \geq 1 - \epsilon,$$

By (7.3) and definition of the approximate tangent space we can estimate

$$\begin{aligned}
(1 - \epsilon)\Theta^{\mathbf{n}}(\mu_s, y) &\leq \Theta^{\mathbf{n}}(\mu_s, y) \int_{\mathbf{T}(\mu_s, y)} \Phi_{(0,0)}(-1, x) \zeta(x) \, d\mathcal{H}^{\mathbf{n}}(x) \\
&\leq \lim_{\lambda \searrow 0} \lambda^{-\mathbf{n}} \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(0,0)}(-1, \lambda^{-1}(x - y)) \, d\mu_s(x) \\
&= \lim_{\lambda \searrow 0} \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(s+\lambda^2, y)}(s, x) \, d\mu_s(x).
\end{aligned}$$

Then with Huisken's monotonicity formula (see Theorem 2.8), and by continuity of the integral we obtain for $h_0 \in (0, s + 1)$ small enough

$$(1 - \epsilon)\Theta^{\mathbf{n}}(\mu_s, y) \leq \lim_{\lambda \searrow 0} \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(s+\lambda^2, y)} \, d\mu_{s-h_0} \leq \lim_{h \searrow 0} \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(s, y)} \, d\mu_{s-h} + \epsilon,$$

Thus by (7.2) and as ϵ was arbitrary we have

$$(7.4) \quad 1 \leq \Theta^{\mathbf{n}}(\mu_s, y) \leq \lim_{h \searrow 0} \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(s, y)} \, d\mu_{s-h} \leq 1$$

for all $y \in D(s)$ for all $s \in (-1, 0)$. Hence μ_t has unit density for a.e. $t \in (-1, 0)$.

Fix an arbitrary $t_0 \in (-1, 0)$ such that μ_{t_0} has unit density. Assumption $\text{spt}\mu_0 \neq \emptyset$ and Lemma 2.6 imply $\text{spt}\mu_{t_0} \neq \emptyset$, so we can find $\mathbf{n} + 1$ points $y_0, \dots, y_{\mathbf{n}}$ in $D(t_0)$ such that $v_i := y_i - y_0$, $i = 1, \dots, \mathbf{n}$ are linearly independent. Set $T := \text{span}(v_i)_{1 \leq i \leq \mathbf{n}}$. By estimates (7.2), (7.4) and Theorem 2.8 we obtain

$$\int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(t_0, y_i)} \, d\mu_t = 1$$

for all $t \in [-1, t_0)$ for all $i \in \{0, \dots, \mathbf{n}\}$.

Then Theorem 2.8 yields the existence of a $J \subset (-1, t_0)$ such that for all $t \in J$ we have μ_t has unit density, the generalised mean curvature vector \mathbf{H}_{μ_t} exists with $\int |\mathbf{H}_{\mu_t}|^2 \mu_t < \infty$ and

$$(7.5) \quad \mathbf{H}_{\mu_t}(x) + (2(t_0 - t))^{-1}(\mathbf{T}(\mu_t, x)^\perp)_{\mathfrak{h}}(x - y_i) = 0$$

for μ_t -a.e. $x \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and all $i = 0, 1, \dots, \mathbf{n}$. Moreover $\mathcal{L}^1((-1, t_0) \setminus J) = 0$.

Let $t \in J$ and let E_t be the set of points $x \in \text{spt}\mu_t$ such that $\Theta^{\mathbf{n}}(\mu_t, x) \geq 1$, $\mathbf{T}(\mu_t, x)$ exists and (7.5) holds for all $i \in \{0, \dots, \mathbf{n}\}$. We see $\mu_t(\mathbb{R}^{\mathbf{n}+\mathbf{k}} \setminus E_t) = 0$. Consider $x \in E_t$ then by (7.5) we have

$$(\mathbf{T}(\mu_t, x)^\perp)_{\mathfrak{h}}(y_0 - y_i) = (\mathbf{T}(\mu_t, x)^\perp)_{\mathfrak{h}}(x - y_i) - (\mathbf{T}(\mu_t, x)^\perp)_{\mathfrak{h}}(x - y_0) = 0$$

for all $i \in \{1, \dots, \mathbf{n}\}$. So $v_i = y_i - y_0 \in \mathbf{T}(\mu_t, x)$ for all $i \in \{1, \dots, \mathbf{n}\}$, hence $\mathbf{T}(\mu_t, x) = \text{span}(v_i)_{1 \leq i \leq \mathbf{n}} = T$. As this holds for all $x \in E_t$ for all $t \in J$, we have $\mathbf{H}_{\mu_t} \equiv 0$ for a.e. $t \in (-1, t_0)$. This follows from Brakke's general regularity theorem [Bra78, Thm. 6.12] (see also [KT14, Thm. 3.2]). Or deduce this from Menne's characterization of the mean curvature vector [Men15, Thm. 15.6].

Now for a.e. $t \in (-1, t_0)$ we have $t \in J$ and $\mathbf{H}_{\mu_t} \equiv 0$, so equality (7.5) with $i = 0$ yields $E_t \subset T + y_0$. Thus $\text{spt}\mu_t \subset T + y_0$. Then by Theorem 1.5 we have $\mu_t = \mathcal{H}^{\mathbf{n}} \llcorner (T + y_0)$. As this holds for a.e. t in $(-1, t_0)$ and by the continuity properties of the Brakke flow due to Brakke [Bra78, Thm. 3.10] we obtain $\mu_t = \mathcal{H}^{\mathbf{n}} \llcorner (T + y_0)$ for all $t \in (-1, t_0)$. Finally choose t_0 arbitrary close to 0 to establish the result. \square

Now suppose the Gaussian density ratios are locally bounded by $1 + \delta$. In view of the previous Lemma 7.1 an indirect blow-up argument combined with Ilmanen's compactness theorem (see Theorem 2.11), yields a small neighbourhood in which we have small height and density ratios close to one, see below. In view of Theorem 1.1 this implies Theorem 1.2.

7.2 Lemma. *For all $\epsilon, \sigma \in (0, 1)$ there exists a $\delta \in (0, 1)$ such that the following holds:*

Let $\rho, \rho_0 \in (0, \infty)$ and let $(\mu_t)_{t \in [-\rho^2, 0]}$ be a Brakke flow in $\mathbf{B}(0, 4\sqrt{n}\rho_0)$. Suppose $\rho \leq \rho_0$, $0 \in \text{spt}\mu_0$ and for all $(s, y) \in (-\rho^2, 0] \times \mathbf{B}(0, \rho)$ we have

$$(7.6) \quad \int_{\mathbb{R}^{n+k}} \Phi_{(s,y)} \varphi_{(s,y), \rho_0} d\mu_{-\rho^2} \leq 1 + \delta.$$

Then there exists a $T \in \mathbf{G}(n+k, n)$ such that we have

$$\begin{aligned} & \sup \left\{ |(T^\perp)_\sharp(x)|, x \in \text{spt}\mu_{-\sigma\delta^2\rho^2} \cap \mathbf{B}(0, 2\delta\rho) \right\} \leq \epsilon\delta\rho \\ & \text{and } \mu_{-\sigma\delta^2\rho^2}(\mathbf{B}(0, \delta\rho)) \leq \omega_n(1+\epsilon)(\delta\rho)^n. \end{aligned}$$

Proof. We may assume $\rho = \delta^{-1}$. Suppose the statement would be false. Then there exist $\epsilon, \sigma \in (0, 1)$ and for every $j \in \mathbb{N}$ we find an $\rho_j \in [j, \infty)$ and a Brakke flow $(\nu_t^j)_{t \in [-j^2, 0]}$ in $B_j := \mathbf{B}(0, 4\sqrt{n}\rho_j)$ such that $0 \in \text{spt}\nu_0^j$,

$$(7.7) \quad \sup_{t \in [-j^2, s]} \int_{\mathbb{R}^{n+k}} \Phi_{(s,y)} \varphi_{(s,y), \rho_j} d\nu_t^j = \int_{\mathbb{R}^{n+k}} \Phi_{(s,y)} \varphi_{(s,y), \rho_j} d\nu_{-j^2}^j \leq 1 + \frac{1}{j}$$

for all $(s, y) \in (-j^2, 0] \times \mathbf{B}(0, j)$ and one of the following holds

$$(7.8) \quad \inf_{T \in \mathbf{G}(n+k, n)} \sup \left\{ |(T^\perp)_\sharp(x)|, x \in \text{spt}\nu_{-\sigma}^j \cap \mathbf{B}(0, 2) \right\} > \epsilon,$$

$$(7.9) \quad \nu_{-\sigma}^j(\mathbf{B}(0, 1)) > \omega_n(1+\epsilon).$$

Note that the equality in (7.7) follows from Huisken's monotonicity formula (see Theorem 2.8).

To obtain a converging subsequence of the (ν_t^j) we need uniform bounds on the measure of compact sets. We claim that for every $R \in (0, \infty)$ we can find a $D(R) \in (0, \infty)$ such that

$$(7.10) \quad \sup_{j \geq 2} \sup_{t \in [-2, 0]} \nu_t^j(\mathbf{B}(0, R) \cap \mathbf{B}(0, j/2)) \leq D(R).$$

Before we prove this, we first show

$$(7.11) \quad \sup_{j \geq 2} \sup_{t \in [-2, 0]} \sup_{y \in \mathbf{B}(0, j)} \sup_{R \in (0, j/4]} \nu_t^j(\mathbf{B}(y, R)) \leq CR^n$$

To see (7.11) consider $R \leq j/4$, $x \in \mathbf{B}(y, 2R)$, $t \in [-2, 0]$ and $c_2 := (2n)^{-1}$. Then we can estimate $\Phi_{(t,y)}(t - c_2 R^2, x) \geq (4\pi c_2 R^2)^{-\frac{n}{2}} \exp(-1/c_2)$ as well as $\varphi_{(t,y), \rho_j}(t - c_2 R^2, x) \geq (1 - 1/4)^3$. Thus Lemma 2.6 and assumption (7.7) yield

$$\begin{aligned} \nu_t^j(\mathbf{B}(y, R)) & \leq C \nu_{t-c_2 R^2}^j(\mathbf{B}(y, 2R)) \\ & \leq CR^n \int_{\mathbb{R}^{n+k}} \Phi_{(t,y)} \varphi_{(t,y), \rho_j} d\nu_{t-c_2 R^2}^j \leq CR^n. \end{aligned}$$

To prove (7.10) let $R \in (0, \infty)$. For $j \leq 4R$ we can use Lemma 2.6 to estimate

$$\nu_t^j(\mathbf{B}(0, j/2)) \leq \nu_t^j(\mathbf{B}(0, 4\sqrt{\mathbf{n}}j)) \leq \nu_{-2}^j(\mathbf{B}(0, 8\sqrt{\mathbf{n}}j)) =: D_j(R)$$

for all $t \in [-2, 0]$. Combined with (7.11) this proves (7.10).

Now we can use the compactness theorem by Ilmanen, (see Theorem 2.11) with $U_j = \mathbf{B}(0, j/2)$, to see that a subsequence of the (ν_t^j) converges to a Brakke flow $(\nu_t)_{t \in [-2, 0]}$ in $\mathbb{R}^{\mathbf{n}+\mathbf{k}}$. Note that without loss of generality we will assume that the whole sequence converges. In particular

$$(7.12) \quad \nu_t(\phi) = \lim_{j \rightarrow \infty} \nu_t^j(\phi) \quad \text{for all } \phi \in \mathcal{C}_c^0(\mathbf{B}(0, j_0/2))$$

for all $t \in [-2, 0]$ and all $j_0 \in \mathbb{N}$. Combining this with (7.11) yields

$$(7.13) \quad \sup_{t \in [-2, 0]} \sup_{y \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}} \sup_{R \in (0, \infty)} \nu_t(\mathbf{B}(y, R)) \leq CR^{\mathbf{n}}.$$

Next we want to show

$$(7.14) \quad \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(s,y)} \, d\nu_t \leq 1$$

for all $(s, y) \in (-2, 0] \times \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and all $t \in [-2, s]$. To see this fix s, y , and t like that. First we see that by (7.13) we have

$$(7.15) \quad \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(s,y)} \, d\nu_t < \infty.$$

In order to prove (7.15) consider $f_l : \mathbb{R}^{\mathbf{n}+\mathbf{k}} \rightarrow \mathbb{R}^+$ given by $f_l(x) := \Phi_{(s,y)}(t, x)$ for $|x - y| < l$ and $f_l \equiv 0$ outside $\mathbf{B}(y, l)$. Obviously we have $f_{l+1} \geq f_l$. Use (7.13) to estimate $\nu_t(\mathbf{B}(y, 2l)) \leq 2C_1(2l)^{\mathbf{n}}$ for all $l \in \mathbb{N}$. Then for $l \geq l_0$ we observe

$$\begin{aligned} \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} f_{l+1} \, d\nu_t - \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} f_l \, d\nu_t &\leq \int_{\mathbf{B}(y, l+1) \setminus \mathbf{B}(y, l)} \Phi_{(s,y)} \, d\nu_t \\ &\leq C(s-t)^{-\mathbf{n}/2} \exp(-l^2/(4(s-t))) \nu_t(\mathbf{B}(y, 2l)) \leq l^{-3-\mathbf{n}} \nu_t(\mathbf{B}(y, 2l)) \leq l^{-2}, \end{aligned}$$

where we chose l_0 large enough depending on $s - t$. Thus $\lim_{l \rightarrow \infty} \int f_l d\nu_t < \infty$ and the monotone convergence theorem implies (7.15).

We continue to prove (7.14). Let $\gamma \in (0, 1)$ be arbitrary. Note that y, s, t are still fixed. Using (7.15) and (7.12) we find $j_1, j_2, j_3 \in \mathbb{N}$, $j_3 > j_2 > j_1$ such that $\rho_{j_2} \geq \rho_{j_1} + 1$,

$$\begin{aligned} \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}} \setminus \mathbf{B}(y, \rho_{j_1})} \Phi_{(s,y)} \, d\nu_t &\leq \gamma, \\ \int_{\mathbf{B}(y, \rho_{j_1})} \Phi_{(s,y)} \, d\nu_t - \int_{\mathbf{B}(y, \rho_{j_2})} \Phi_{(s,y)} \, d\nu_t^j &\leq \gamma, \\ 1 &\leq \inf_{x \in \mathbf{B}(y, \rho_{j_2})} \varphi_{(s,y), \rho_j}(t, x) + \gamma \rho_{j_2}^{-\mathbf{n}} \end{aligned}$$

for all $j \geq j_3$. Combining these estimates with (7.11) we obtain

$$\int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(s,y)} \, d\nu_t \leq \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(s,y)} \varphi_{(s,y), \rho_j} \, d\nu_t^j + (2 + C(s-t)^{-\mathbf{n}/2}) \gamma$$

for all $j \geq j_3$. By (7.7) and as s, t, y, γ were arbitrary this establishes (7.14).

In view of (7.13) and (7.14) we can use Lemma 7.1 to obtain a subspace $T \in \mathbf{G}(\mathbf{n} + \mathbf{k}, \mathbf{n})$ such that

$$(7.16) \quad \nu_t = \mathcal{H}^{\mathbf{n}} \lrcorner T$$

for all $t \in (-2, 0)$. Note that by (7.12) and as $0 \in \text{spt} \nu_0^j$ for all $j \in \mathbb{N}$ we see $a = 0$ is a porpper choice in Lemma 7.1. This should now contradict that (7.8) or (7.9) hold for infinitely many j .

First suppose that for infinitely many j inequality (7.8) holds, i.e. there exists a $z_j \in \text{spt} \nu_{-\sigma}^j \cap \mathbf{B}(0, 1)$ such that $(T^\perp)_{\mathfrak{h}}(z_j) > \epsilon$. Consider C_1 and α_1 from the clearing out lemma (see Lemma 2.9). Choose $\tau, \eta_1 \in (0, 1)$ such that $4\mathbf{n}\tau < (\epsilon/4)^2$ and $C_1\eta_1^{2\alpha_1}(\epsilon/4)^2 \leq \tau$. Then Lemma 2.9 with $R = \epsilon/4$ yields that $\nu_{-\sigma-\tau}^j(\mathbf{B}(z_j, \epsilon/4)) > \eta_1$ for infinitely many j . A subsequence of the z_j converges to some $z_0 \in \mathbf{B}(0, 2)$ with $(T^\perp)_{\mathfrak{h}}(z_0) \geq \epsilon$. Consider a cut-off function $\zeta_1 \in \mathcal{C}_c^\infty(\mathbf{B}(z_0, \epsilon/2), [0, 1])$ with $\{\zeta_1 = 1\} \supset \mathbf{B}(z_0, \epsilon/3)$. Then

$$\nu_{-\sigma-\tau}^j(\zeta_1) \geq \nu_{-\sigma-\tau}^j(\mathbf{B}(z_j, \epsilon/4)) > \eta_1 > 0 = \nu_{-\sigma-\tau}(\zeta_1)$$

for infinitely many j , where we used (7.16) for the last equality. In view of (7.12) this yields a contradiction.

Now suppose that for infinitely many j inequality (7.9) holds. Consider $\zeta_2 \in \mathcal{C}_c^\infty(\mathbf{B}(0, \sqrt[3]{1 + \epsilon/2}), [0, 1])$ with $\{\zeta_2 = 1\} \supset \mathbf{B}(0, 1)$. In view of (7.16) we can estimate

$$\nu_{-\sigma}(\zeta_2) \leq \omega_{\mathbf{n}}(1 + \epsilon/2) < \omega_{\mathbf{n}}(1 + \epsilon) \leq \nu_{-\sigma}^j(\mathbf{B}(0, 1)) \leq \nu_{-\sigma}^j(\zeta_2)$$

for infinitely many j . Again, this contradicts (7.12), which establishes the result. \square

Proof of Theorem 1.2. We may assume $t_0 = 0$, $a = 0$ and $\rho = 1$. Let α_0 and γ_0 be from Theorem 1.1 with respect to $\lambda = 1/2$. Choose $\epsilon \in (0, \gamma_0]$ such that $2\epsilon^{\alpha_0} \leq \alpha_0\beta$ and set $\sigma := 2\epsilon^{\alpha_0}$. Let δ be chosen with respect to ϵ and σ according to Lemma 7.2 and choose $\eta \leq \delta$, $t_1 := -2\epsilon^{\alpha_0}\delta^2$. Then Lemma 7.2 with yields the existence of a $T \in \mathbf{G}(\mathbf{n} + \mathbf{k}, \mathbf{n})$ such that

$$\begin{aligned} \sup \{ |(T^\perp)_{\mathfrak{h}}(x)|, x \in \text{spt} \mu_{t_1} \cap \mathbf{B}(0, 2\delta) \} &\leq \epsilon\delta. \\ \mu_{t_1}(\mathbf{B}(0, \delta)) &\leq \omega_{\mathbf{n}}(1 + \epsilon)\delta^{\mathbf{n}}. \end{aligned}$$

Then Theorem 1.1 with $\rho = \delta$ and $\gamma = \epsilon$ yields the desired graphical representation for $\eta = \epsilon^{\alpha_0}\delta$. \square

A Curvature estimates

A.1 Setting. Consider open and connected sets $W \subset \mathbb{R}^{\mathbf{n}}$, $J \subset \mathbb{R}$. Consider $\Psi \in \mathcal{C}^\infty(J \times W, \mathbb{R}^{\mathbf{n}+\mathbf{k}})$ such that $\Psi_t := \Psi(t, \cdot)$ are embeddings and set $M_t := \Psi_t[W]$ for all $t \in J$. Fix $t \in J$ and $\hat{y} \in W$. We set

$$\begin{aligned} b_i(t, \hat{y}) &:= D_i \Psi(t, \hat{y}), \quad g_{ij}(t, \hat{y}) := b_i(t, \hat{y}) \cdot b_j(t, \hat{y}), \\ \mathbf{T}(M_t, \Psi(t, \hat{y})) &:= \text{span}(b_i(t, \hat{y}))_{1 \leq i \leq \mathbf{n}} \\ A_{ij}(t, \hat{y}) &:= (\mathbf{T}(M_t, \Psi(t, \hat{y}))^\perp)_{\mathfrak{h}} D_i D_j \Psi(t, \hat{y}) \in \mathbb{R}^{\mathbf{n}+\mathbf{k}} \end{aligned}$$

for $i, j \in \{1, \dots, \mathbf{n}\}$. In the following we sum over repeated indices. Let $(g^{ij}(t, \hat{y}))$ be the inverse to $(g_{ij}(t, \hat{y}))$. and let $(db^i(t, \hat{y}))_{1 \leq i \leq \mathbf{n}}$ be the dual basis to $(b_i(t, \hat{y}))_{1 \leq i \leq \mathbf{n}}$. We set

$$\begin{aligned}\mathbf{A}(M_t, \Psi_t(\hat{y})) &:= A(t, \hat{y}) := A_{ij}(t, \hat{y}) db^i(t, \hat{y}) \otimes db^j(t, \hat{y}) \\ \mathbf{H}(M_t, \Psi_t(\hat{y})) &:= g^{ij}(t, \hat{y}) A_{ij}(t, \hat{y}).\end{aligned}$$

Consider $T = T_{i_1 \dots i_p} db^{i_1} \otimes \dots \otimes db^{i_p}$ and $S = S_{j_1 \dots j_p} db^{j_1} \otimes \dots \otimes db^{j_p}$ with $T_{i_1 \dots i_p}, S_{j_1 \dots j_p} \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$. We set

$$\langle T, S \rangle := g^{i_1 j_1} \dots g^{i_p j_p} T_{i_1 \dots i_p} \cdot S_{j_1 \dots j_p}, \quad |T|^2 := \langle T, T \rangle.$$

Moreover we assume Ψ moves by smooth mean curvature flow in normal direction, i.e.

$$(A.1) \quad \partial_t \Psi(t, \hat{y}) = \mathbf{H}(M_t, \Psi_t(\hat{y}))$$

for all $(t, \hat{y}) \in J \times W$.

A.2 Lemma. *Consider open and connected sets $W_0 \subset\subset W \subset\subset \Omega \subset \mathbb{R}^{\mathbf{n}}$, $I \subset \mathbb{R}$. Let $F \in C^\infty(I \times \Omega, \mathbb{R}^{\mathbf{n}+\mathbf{k}})$ be such that $F_t := F(t, \cdot)$ are embeddings and set $M_t := F_t[\Omega]$ for all $t \in I$. Suppose F satisfies (1.1) and fix $s_0 \in I$.*

Then there exists an open $J \subset I$ with $s_0 \in J$ and a $\phi \in C^\infty(J \times W, \mathbb{R}^{\mathbf{n}})$ such that $\phi_t := \phi(t, \cdot)$ is injective and $W_0 \subset \phi_t[W]$ for all $t \in J$. Furthermore ϕ satisfies $\phi_{s_0}(\hat{y}) = \hat{y}$ for all $\hat{y} \in W$ and

$$(A.2) \quad DF(t, \phi(t, \hat{y})) \partial_t \phi(t, \hat{y}) = -(\mathbf{T}(M_t, F(t, \phi(t, \hat{y}))))_{\sharp} \partial_t F(t, \phi(t, \hat{y}))$$

for all $(t, \hat{y}) \in J \times W$. In particular $\Psi(t, \hat{y}) := F(t, \phi(t, \hat{y}))$ satisfies (A.1) and we are in Setting A.1.

Proof. Actually (A.2) equals

$$(A.3) \quad \partial_t \phi(t, \hat{y}) = G(t, \phi(t, \hat{y}))$$

for some $G \in C^\infty(I \times \Omega, \mathbb{R}^{\mathbf{n}})$ depending on F . The function G can be obtained in the following way: Let $p_i := D_i F$, $B_{ij} := p_i \cdot p_j$ and (B^{ij}) be the inverse of (B_{ij}) . Set $q_i := B^{ij} p_j$. Now we obtain (A.3) from

$$Q := (q_1 \dots q_{\mathbf{n}}), \quad DF = (p_1 \dots p_{\mathbf{n}}), \quad Q^T v = (v \cdot p_i) \hat{\mathbf{e}}_i, \quad G := -Q^T \partial_t F,$$

for all $v \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$. Then the Picard-Lindelöf-theorem yields the result. \square

Next we need a bound on the evolution of the absolute value of the curvature tensor and its covariant derivatives. In the case of one co-dimension this was done by Huisken [Hui84, Thm. 3.4] (see also [Ham82, Sect. 13]). Here we state the generalization by Andrews and Baker.

A.3 Proposition ([AB10, Prop. 9]). *For every $m \in \mathbb{N} \cup \{0\}$ there exists an $C_m \in [1, \infty)$ such that in Setting A.1 the following holds:*

$$\partial_t |\nabla^m A|^2 \leq \Delta |A|^2 - 2 |\nabla^{m+1} A|^2 + C_m \sum_{a+b+c=m} |\nabla^m A| |\nabla^a A| |\nabla^b A| |\nabla^c A|$$

everywhere in $J \times W$. Here ∇ and Δ are the connection and Laplace-Beltrami operator associated to $(M_t, (g_{ij}))$.

Now we can prove a higher regularity theorem similar to Ecker's [Eck04, Prop. 3.22] (see also [EH91]).

A.4 Proposition. *For all $m \in \mathbb{N}$ and $K \in [1, \infty)$ there exists a $C_{m,K} \in (1, \infty)$ such that the following holds:*

Consider an open and connected set $\Omega \subset \mathbb{R}^n$. Let $\rho \in (0, \infty)$, $t_0 \in \mathbb{R}$, $a \in \mathbb{R}^{n+k}$ and let $F \in \mathcal{C}^2((t_0 - 4\rho^2, t_0) \times \Omega, \mathbb{R}^{n+k})$ be such that for all $t \in (t_0 - 4\rho^2, t_2)$ we have $F_t := F(t, \cdot)$ are embeddings, $M_t := F_t[\Omega]$ and $\partial M_t \cap \mathbf{B}(a, 2\rho) = \emptyset$. Suppose F satisfies (1.1) and

$$\kappa^2 := \rho^2 \sup_{t \in (t_0 - 4\rho^2, t_0)} \sup_{x \in M_t \cap \mathbf{B}(a, 2\rho)} |\mathbf{A}(M_t, x)|^2 \leq K$$

Then we have

$$\rho^{2m+2} \sup_{t \in [t_0 - \rho^2, t_0]} \sup_{x \in M_t \cap \mathbf{B}(a, \rho)} |\nabla^m \mathbf{A}(M_t, x)|^2 \leq C_{m,K} \kappa^2$$

Proof. Obviously the statement holds for $m = 0$. Now suppose the statement holds for all $l = 0, \dots, m$ for some $m \in \mathbb{N} \cup \{0\}$ and show this implies it also holds for $m + 1$. We may assume $t_0 = 0$, $a = 0$ and $\rho = 3$. Fix $t_2 \in (-9, 0)$ and let $\Lambda \in [1, \infty)$ be chosen below. Consider the family of functions $\alpha_t : M_t \rightarrow \mathbb{R}^+$, $t \in (-12, 0)$ given by

$$\alpha_t(x) := |\nabla^{m+1} \mathbf{A}(M_t, x)|^2 (\Lambda \kappa^2 + |\nabla^m \mathbf{A}(M_t, x)|^2).$$

Also consider the cut-off functions $\eta_t \in \mathcal{C}_c^2(\mathbf{B}(0, 5), \mathbb{R}^+)$, $t \in (-12, 0)$ given by

$$\eta_t(x) := ((t + 10)_+ (16 - |x|^2)_+)^3.$$

For $t \in (-12, 0)$ set

$$m(t) := \sup_{M_t} \alpha_t \eta_t, \quad m_0 := \sup_{t \in [-10, t_2]} m(t), \quad s_0 := \inf\{t \in [t_1, t_2], m(t) = m_0\}.$$

There exists an $x_0 \in M_{s_0}$ such that $\alpha_{s_0}(x_0) \eta_{s_0}(x_0) = m_0$ and $\hat{y}_0 \in \Omega$ with $F_{s_0}(\hat{y}_0) = x_0$. Consider $s_3 = \min\{s_0 + 1/2, s_0/2\}$ in particular $s_0 \in (s_3 - 1, s_3)$. By induction assumption we can use the Proposition for m with a , t_0 , ρ replaced by x_0 , s_3 , 1 to obtain

$$(A.4) \quad \max_{l=1, \dots, m} |\nabla^l \mathbf{A}(M_{s_0}, x_0)|^2 \leq \Gamma_m \kappa^2$$

for some $\Gamma_m \in (K, \infty)$ depending only on m and K . In the following quantities that only depend on m , K and Γ_m will be denoted by C_m .

In view of Lemma A.2 there exists an $\epsilon \in (0, -s_0)$, $J = (s_0 - \epsilon, s_0 + \epsilon)$, an open $W \subset \Omega$ and a $\phi \in \mathcal{C}^\infty(J \times W, \Omega)$, such that $\phi_t = \phi(t, \cdot)$ is a diffeomorphism onto its image and $F^{-1}[M_t \cap \mathbf{B}(x_0, \epsilon)] \subset \phi_t[W]$ for all $t \in J$. Moreover choose ϕ such that $\Psi_t(\hat{y}) := F_t(\phi_t(\hat{y}))$ satisfies (A.1). In particular $\hat{y}_0 \in W$ and we are in Setting A.1. Consider $\hat{\alpha} \in \mathcal{C}^\infty(J \times W, \mathbb{R})$ given by

$$\hat{\alpha}(t, \hat{x}) := \hat{\alpha}_t(\hat{x}) := \alpha_t(\Psi_t(\hat{x})) = |\nabla^{m+1} A(t, \hat{x})|^2 (\Lambda \kappa^2 + |\nabla^m A(t, \hat{x})|^2)$$

in the Ψ coordinates. Using Proposition A.3 and inequality (A.4) we can estimate at (s_0, \hat{y}_0)

$$\begin{aligned} (\partial_t - \Delta)\hat{\alpha} &= (\Lambda\kappa^2 + |\nabla^m A|^2)(\partial_t - \Delta)|\nabla^{m+1} A|^2 + |\nabla^{m+1} A|^2(\partial_t - \Delta)|\nabla^m A|^2 \\ &\quad - 2\nabla|\nabla^{m+1} A|^2 \cdot \nabla|\nabla^m A|^2 \\ &\leq \Lambda\kappa^2 (-2|\nabla^{m+2} A|^2 + C_m|\nabla^{m+1} A|(|\nabla^{m+1} A|\kappa^2 + \kappa^3)) \\ &\quad - 2|\nabla^{m+1} A|^4 + C_m|\nabla^{m+1} A|^2\kappa^4 - 2\nabla|\nabla^{m+1} A|^2 \cdot \nabla|\nabla^m A|^2. \end{aligned}$$

By Young's and Kato's inequality we can estimate

$$\begin{aligned} C_m\Lambda|\nabla^{m+1} A|\kappa^5 &\leq C_m\Lambda|\nabla^{m+1} A|^2\kappa^4 + \kappa^6, \\ C_m\Lambda|\nabla^{m+1} A|^2\kappa^4 &\leq |\nabla^{m+1} A|^4/2 + C_m\Lambda^2\kappa^8, \\ 2|\nabla|\nabla^{m+1} A|^2 \cdot \nabla|\nabla^m A|^2| &\leq 8|\nabla^m A||\nabla^{m+1} A|^2|\nabla^{m+2} A| \\ &\leq \Lambda\kappa^2|\nabla^{m+2} A|^2 + 2\Lambda^{-1}|\nabla^{m+1} A|^4. \end{aligned}$$

Fixing $\Lambda = 4$ this leads to

$$(\partial_t - \Delta)\hat{\alpha} \leq -|\nabla^{m+1} A|^4 + C_m\kappa^4 \leq -L^{-1}\kappa^{-4}\hat{\alpha}^2 + L\kappa^4$$

at (s_0, \hat{y}_0) for some $L \in [1, \infty)$ depending only on m , K and Γ_m . Consider $\hat{\eta}(t, \hat{x}) := \eta_t(\Psi(t, \hat{x}))$ then at (s_0, \hat{y}_0) we can estimate

$$\begin{aligned} (\partial_t - \Delta)(\hat{\alpha}\hat{\eta}) &\leq \hat{\eta}(L\kappa^4 - L^{-1}\kappa^{-4}\hat{\alpha}^2) + C\hat{\alpha} - 2\nabla\hat{\alpha} \cdot \nabla\hat{\eta} \\ &\leq \hat{\eta}(L\kappa^4 - L^{-1}\kappa^{-4}\hat{\alpha}^2) + C\hat{\alpha} - 2\hat{\eta}^{-1}\nabla\hat{\eta} \cdot \nabla(\hat{\alpha}\hat{\eta}). \end{aligned}$$

By choice of s_0 and \hat{y}_0 we have $\Delta(\hat{\alpha}\hat{\eta}) \leq 0$, $\nabla(\hat{\alpha}\hat{\eta}) = 0$ and $\partial_t(\hat{\alpha}\hat{\eta}) \geq 0$. Hence

$$(L\kappa^4)^{-1}\hat{\alpha}^2\hat{\eta} \leq L\kappa^4\hat{\eta} + C\hat{\alpha}$$

at (s_0, \hat{y}_0) . Multiplying with $\hat{\eta}$ and using Young's inequality we obtain

$$(L\kappa^4)^{-1}(\hat{\alpha}\hat{\eta})^2 \leq L\kappa^4\hat{\eta}^2 + C\hat{\alpha}\hat{\eta} \leq C_m\kappa^4 + (2L\kappa^4)^{-1}(\hat{\alpha}\hat{\eta})^2$$

at (s_0, \hat{y}_0) . Thus $m_0 = \hat{\alpha}(s_0, \hat{y}_0)\hat{\eta}(s_0, \hat{y}_0) \leq C_m\kappa^4$ and we arrive at

$$|\nabla^{m+1} \mathbf{A}(M_t, x)|^2(\Lambda\kappa^2 + |\nabla^m \mathbf{A}(M_t, x)|^2)\eta_t(x) \leq C_m\kappa^4$$

for all $t \in [-10, t_2]$ and all $x \in M_t$. For $t \in [-9, t_2]$ and $x \in M_t \cap \mathbf{B}(0, 3)$ we have $\eta_t(x) \geq 1$, so we can conclude the desired estimate on $[-9, t_2]$. As $t_2 \in [-9, 0)$ was arbitrary this establishes the statement for $m+1$. \square

The above curvature estimates will be used in the form of the next Lemma. It basically says that if you have a smooth mean curvature flow that satisfies a strong curvature bound up to some time t_1 and the curvature on $[t_1, t_2]$ is a-priori bounded by some constant, then the increase in curvature is controlled by $(t_2 - t_1)^p$. Thus for short times you can actually almost maintain the strong curvature bound.

A.5 Lemma. *For $p \in \mathbb{N}$ and $L \in [1, \infty)$ there exists a $C_{p,L} \in (1, \infty)$ such that the following holds:*

Consider an open and connected set Ω . Let $\varrho \in (0, \infty)$, $z_0 \in \mathbb{R}^{n+k}$, $t_2 \in \mathbb{R}$, $t_1 \in [t_2 - \varrho^2/(4pL), t_2]$ and let $F \in \mathcal{C}^2((t_2 - \varrho^2, t_2) \times \Omega, \mathbb{R}^{n+k})$ be such that for all $t \in (t_2 - \varrho^2, t_2)$ we have $F_t := F(t, \cdot)$ are embeddings, $M_t := F_t[\Omega]$ and $\partial M_t \cap \mathbf{B}(z_0, 2\varrho) = \emptyset$. Suppose F satisfies (1.1). For $s \in [t_1, t_2]$, $r \in (0, 2\varrho]$ set

$$\kappa(s, r)^2 := \varrho^2 \sup_{t \in (t_2 - \varrho^2, s)} \sup_{x \in M_t \cap \mathbf{B}(z_0, r)} |\mathbf{A}(M_t, x)|^2$$

and assume $\kappa(t_2, 2\varrho) \leq L$.

Then we have

$$\kappa(t_2, \varrho)^2 \leq C_{p,L} (\kappa(t_1, 2\varrho)^2 + \varrho^{-2p} (t_2 - t_1)^p).$$

Proof. We may assume $z_0 = 0$, $t_1 = 0$ and $\varrho = 1$. Consider $q = 0, 1, \dots, p$ and set $\varrho_q := 2 - q/p$. We will actually show

$$(A.5) \quad \kappa(t_2, \varrho_q)^2 \leq \Gamma_q (2\kappa(0, 2)^2 + t_2^q).$$

for some $\Gamma_q \in [1, \infty)$ depending only on L , p and q . For $q = 0$ this holds with $\Gamma_0 = L$. Now suppose (A.5) holds for some $q \in \{0, 1, \dots, p-1\}$. For $t \in [0, t_2]$ set $\varrho_q(t) := \varrho_q - 1/(2p) - tL$. We want to show

$$(A.6) \quad \kappa(t, \varrho_q(t))^2 \leq \Gamma_{q+1} ((t+1)\kappa(0, 2)^2 + t t_2^q).$$

for some $\Gamma_{q+1} \in [1, \infty)$ depending only on L , p , q and Γ_q . Let I_0 be the interval containing all $s \in [0, t_2]$ such that (A.6) holds. Note that by $t_2 \leq 1/(4pL)$ we have $\varrho_q(t_2) \geq \varrho_{q+1}$, thus if $t_2 \in I_0$ we can conclude the result by induction. Obviously $0 \in I_0$. Also, for an increasing sequence (s_m) in I_0 with $s_m \nearrow s_0$ the smoothness of (M_t) yields that $s_0 \in I_0$. To establish $I_0 = [0, t_2]$ (and thus the result) it remains to show that for each $s_0 \in I_0$ there exists an $\epsilon \in (0, t_2 - s_0)$ such that $(s_0, s_0 + \epsilon) \subset I_0$. Hence, consider $s_0 \in I_0$.

In view of Lemma A.2 there exists an $\epsilon \in (0, t_2 - s_0)$, $J = (s_0 - \epsilon, s_0 + \epsilon)$, an open $W \subset \Omega$ and $\phi \in \mathcal{C}^\infty(J \times W, \Omega)$, such that $\phi_t = \phi(t, \cdot)$ is a diffeomorphism onto its image and $F^{-1}[M_t \cap \mathbf{B}(0, 2)] \subset \phi_t[W]$ for all $t \in J$. Moreover choose ϕ such that $\Psi_t(\hat{y}) := F_t(\phi_t(\hat{y}))$ satisfies (A.1), in particular we are in Setting A.1.

Consider an $s_2 \in (s_0, s_0 + \epsilon)$ and $x_0 \in M_{s_2} \cap \mathbf{B}(0, \rho_q(s_2))$, hence there exists an $\hat{y}_0 \in W$ such that $x_0 = \Psi_{s_2}(\hat{y}_0)$. By (A.1) and $\kappa(t_2, 2) \leq L$ we have

$$|\Psi_s(\hat{y}_0)| \leq |x_0| + \int_{s_0}^s |D_t \Psi_t(\hat{y}_0)| dt < \rho_q(s_2) + (s_2 - s_0)L \leq \rho_q(0)$$

for all $s \in [s_0, s_2]$. By Proposition A.3 we have

$$(A.7) \quad \left| \frac{d}{ds} |\mathbf{A}(M_s, \Psi_s(\hat{y}_0))|^2 \right| \leq C_1 \sum_{i=0}^2 \sup_{t \in [0, s_2]} \sup_{x \in M_t \cap \mathbf{B}(0, \rho_q(0))} |\nabla^i \mathbf{A}(M_t, x)|^2$$

for $s \in (s_0, s_2)$ and some constant $C_1 \in [1, \infty)$. We assume (A.5) holds for q , so by $\rho_q(0) \leq \rho_q - 1/(2p)$ and Proposition A.4 with $\rho = 1/(4p)$, $K = \Gamma_q(2L^2 + 1)$ and arbitrary $(t_0, a) \in [0, s_2] \times \mathbf{B}(0, \rho_q(0))$ we have

$$(A.8) \quad \sum_{i=0}^2 \sup_{t \in [0, s_2]} \sup_{x \in M_t \cap \mathbf{B}(0, \rho_q(0))} |\nabla^i \mathbf{A}(M_t, x)|^2 \leq \Lambda_q (\kappa(0, 2)^2 + t_2^q).$$

for some $\Lambda_q \in [1, \infty)$ depending only on L, p, q and Γ_q .

As $s_0 \in I_0$ and $|\Psi_{s_0}(\hat{y}_0)| < \rho_q(0) \leq \rho_q(s_0)$ inequality (A.6) combined with (A.7) and (A.8) yield

$$\begin{aligned} |\mathbf{A}(M_{s_2}, x_0)|^2 &\leq |\mathbf{A}(M_{s_0}, \Psi_{s_0}(\hat{y}_0))|^2 + (s_2 - s_0) \sup_{t \in [s_0, s_2]} \left| \frac{d}{dt} |\mathbf{A}(M_t, \Psi_t(\hat{y}_0))|^2 \right| \\ &\leq \Gamma_{q+1}((s_0 + 1)\kappa(0, 2)^2 + s_0 t_2^q) + C_1 \Lambda_q (s_2 - s_0)(\kappa(0, 2)^2 + t_2^q) \\ &\leq \Gamma_{q+1}((s_2 + 1)\kappa(0, 2)^2 + s_2 t_2^q), \end{aligned}$$

where we chose $\Gamma_{q+1} = C_1 \Lambda_q$. As $s_2 \in (s_0, s_0 + \epsilon)$ and $x_0 \in M_{s_2} \cap \mathbf{B}(0, \rho_q(s_2))$ were arbitrary we proved $(s_0, s_0 + \epsilon) \subset I_0$. \square

For a solution of mean curvature flow, that is graphical with small gradient we obtain an a-priori curvature bound. This follows directly from the work of Wang [Wan04] and Ecker and Huisken [EH91].

A.6 Lemma. *There exist $C \in (1, \infty)$ and $l_1 \in (0, 1)$ with the following property:*

Let $\rho \in (0, \infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1, t_1 + l_1 \rho^2)$, $w \in \mathcal{C}^\infty((t_1, t_2) \times \mathbf{B}^n(0, 2\rho), \mathbb{R}^k)$. Suppose $F_t(\hat{x}) := (t, w(t, \hat{x}))$ satisfies (1.1) and $\sup |Dw| \leq l_1$.

Then we have

$$\sup_{\hat{x} \in \mathbf{B}^n(0, \rho)} |\mathbf{A}(M_t, (\hat{x}, w(t, \hat{x})))|^2 \leq C(t - t_1)^{-1}$$

for all $t \in (t_1, t_2)$, where $M_t = \text{graph}(w(t, \cdot))$.

Proof. Consider $[s_1, s_2] \subset (t_1, t_2)$. We may assume $s_1 = 0$ and $\rho = 1$. For $t \in [0, s_2]$ set $M_t = \text{graph}(w(t, \cdot))$ and $\mathcal{M} := \{(t, x) \in [0, s_2] \times \mathbb{R}^{n+k} : x \in M_t\}$. On \mathcal{M} define

$$*\Omega(t, x) := (\det(\mathbf{Id}_{\mathbb{R}^n} + (Dw(t, \hat{x}))^T Dw(t, \hat{x})))^{-1} \in (0, 1],$$

where $\mathbf{Id}_{\mathbb{R}^n}$ is the identity on \mathbb{R}^n . The gradient bound on w yields

$$(A.9) \quad \inf_{\mathcal{M}} *\Omega \geq (1 + Cl_1)^{-1} \geq 19/20,$$

where we chose l_1 small enough. For $(t, x) \in \mathcal{M}$ consider

$$\begin{aligned} f(t, x) &:= *\Omega(t, x) - 9/10, \quad v(t, x) := 1/f(t, x), \\ \kappa &:= \frac{1}{2} \inf_{t \in [0, s_2]} \inf_{x \in M_t \cap \mathbf{C}(0, \sqrt{2})} (v(t, x))^{-2}, \quad \varphi(t, x) := (v(t, x))^2 / (1 - \kappa(v(t, x))^2), \\ g(t, x) &:= |\mathbf{A}(M_t, x)|^2 \varphi(t, x) \quad \text{and} \quad \eta(x) := (2 - |\hat{x}|^2)^2. \end{aligned}$$

For $t \in [0, s_2]$ set $g_t := g(t, \cdot)$ and

$$m(t) := \sup_{M_t} t g_t \eta, \quad m_0 := \sup_{t \in [0, s_2]} m(t), \quad s_0 := \inf\{t \in [0, s_2], m(t) = m_0\}.$$

Consider $x_0 \in M_{s_0} \cap \mathbf{C}(0, \sqrt{2})$ such that $m_0 = s_0 g(s_0, x_0) \eta(x_0)$.

In view of Lemma A.2 there exists an $\epsilon > 0$, $J = (s_0 - \epsilon, s_0 + \epsilon) \subset (t_1, t_2)$, $W := \mathbf{B}^n(0, 7/4)$ and $\phi \in \mathcal{C}^\infty(J \times W, \Omega)$, such that $\phi_t = \phi(t, \cdot)$ is a diffeomorphism onto its image and $\mathbf{B}^n(0, \sqrt{2}) \subset \phi_t[W]$ for all $t \in J$. Moreover choose ϕ

such that $\Psi_t(\hat{y}) := F_t(\phi_t(\hat{y}))$ satisfies (A.1), in particular we are in Setting A.1. Let $\hat{y}_0 \in W$ be such that $\Psi_{s_0}(\hat{y}_0) = x_0$.

By (A.9) and [Wan04, Lem. 3.1] we can estimate

$$(A.10) \quad (\partial_t - \Delta) \hat{f} = (\partial_t - \Delta) * \Omega \geq |A|^2/2 \geq 5\hat{f}|A|^2$$

everywhere in $J \times W$, where $\hat{f}(t, \hat{x}) := f(\Psi_t(\hat{x}))$. Then by the proof of [Wan04, Cor. 4.1] we obtain

$$(A.11) \quad (\partial_t - \Delta) |A| \leq 5|A|^3$$

everywhere in $J \times W$. Estimates (A.10), (A.11) and the proof of [Wan04, Lem. 4.1] with $h = A$, $c_1 = c_2 = 5$ and ϕ replaced by φ yield

$$(\partial_t - \Delta) \hat{g} \leq -10\kappa\hat{g}^2 - 2\kappa(1 - \kappa\hat{v}^2)^{-2}|\nabla\hat{v}|^2\hat{g} - 2\hat{\varphi}\hat{v}^{-3}\nabla\hat{v} \cdot \nabla\hat{g}$$

everywhere in $J \times W$, where $\hat{g}(t, \hat{x}) := g(t, \Psi_t(\hat{x}))$, $\hat{v}(t, \hat{x}) := v(t, \Psi_t(\hat{x}))$ and $\hat{\varphi}(t, \hat{x}) := \varphi(t, \Psi_t(\hat{x}))$. Then following the proof of [EH91, Thm. 3.1] with $r(t, x) = |x|^2 - |w(t, \hat{x})|$ we see

$$\begin{aligned} (\partial_t - \Delta) \iota\hat{g}\hat{\eta} &\leq -10\kappa\hat{g}^2\hat{\eta}\iota - 2(\hat{\varphi}\hat{v}^{-3}\nabla\hat{v} + \hat{\eta}^{-1}\nabla\hat{\eta}) \cdot \nabla(\iota\hat{g}\hat{\eta}) \\ &\quad + C((1 + (\kappa\hat{v}^2)^{-1}))\iota\hat{g} + \hat{g}\hat{\eta} \end{aligned}$$

everywhere in $J \times W$, where $\hat{\eta}(t, \hat{x}) := \eta(\Psi_t(\hat{x}))$ and $\iota(t, \hat{x}) := t$. In particular at (s_0, x_0) we have

$$10\kappa g^2 \eta s_0 \leq C((1 + (\kappa v^2)^{-1})) s_0 g + g \eta.$$

Recall the definition of κ and that by (A.9) we have $\kappa \geq c$. Also note that $\eta \leq 2$ and $s_0 \leq s_2 \leq l_1 \leq 1$. Thus we obtain

$$s_2 \sup_{M_{s_2} \cap \mathbf{C}(0, \sqrt{2})} g_{s_2} \eta \leq s_0 g(s_0, x_0) \eta(x_0) \leq C.$$

As $\inf_{\mathbf{C}(0,1)} \eta \geq 1$ and $\inf_{\mathcal{M}} \varphi > 1$ we arrive at

$$\sup_{\hat{x} \in \mathbf{B}^n(0,1)} |\mathbf{A}(M_{s_2}, (\hat{x}, w(s_2, \hat{x})))|^2 \leq C(s_2 - s_1)^{-1}$$

for arbitrary $[s_1, s_2] \subset (t_1, t_2)$. Now for any $t \in (t_1, t_2)$ choose $s_2 = t$ and $s_1 = (t + t_1)/2$ to conclude the result. \square

B Further remarks

Based on Huisken's monotonicity formula [Hui90] one can obtain bounds on area ratio at later times from initial area ratio bounds.

B.1 Lemma. *There exists a $C \in (1, \infty)$ such that the following holds:*

Let $r \in (0, \infty)$, $s_1 \in \mathbb{R}$, $s_2 \in (s_1, \infty)$, $y_0 \in \mathbb{R}^{n+k}$ and for $t \in [s_1, s_2]$ set $\varrho(t) := \sqrt{8n} \max\{r, \sqrt{t - s_1}\}$. Let $(\mu_t)_{t \in [s_1, s_2]}$ be a Brakke flow in $\mathbf{B}(y_0, \varrho(s_2))$. Then for all $t \in [s_1, s_2]$ we have

$$r^{-n} \mu_t(\mathbf{B}(y_0, r)) \leq C(\varrho(t))^{-n} \mu_{s_1}(\mathbf{B}(y_0, \varrho(t))).$$

Proof. We may assume $y_0 = 0$, $s_1 = 0$. Fix $t \in [0, s_2]$. First consider the case $t \leq r^2$. Then by Lemma 2.6 with $R = \sqrt{2\mathbf{n}r}$ we have

$$\mu_t(\mathbf{B}(y_0, r)) \leq \mu_t(\mathbf{B}(y_0, \sqrt{2\mathbf{n}r})) \leq 8\mu_0(\mathbf{B}(y_0, 2\sqrt{2\mathbf{n}r}))$$

and as $\varrho(t) = \sqrt{8\mathbf{n}r}$ this yields the desired estimate.

Now suppose $r^2 < t$, in particular $\varrho(t) = \sqrt{8\mathbf{n}t}$. Set $s_0 := t + r^2$. Consider $\Phi = \Phi_{(s_0, 0)}$ and $\varphi = \varphi_{(s_0, 0), \sqrt{4\mathbf{n}t}}$ from Definition 2.7. By $s_0 < 2t$ we obtain

$$\text{spt}\varphi(0, \cdot) \subset \subset \mathbf{B}(0, \varrho(t)), \quad \sup_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \varphi(0, \cdot) \leq C, \quad \inf_{\mathbf{B}(0, r)} \varphi(t, \cdot) \geq 1.$$

Thus by Huiskin's monotonicity formula (see Theorem 2.8)

$$\int_{\mathbf{B}(0, r)} \Phi \, d\mu_t \leq \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi \varphi \, d\mu_t \leq \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi \varphi \, d\mu_0 \leq C \int_{\mathbf{B}(0, \varrho(t))} \Phi \, d\mu_0.$$

We have $s_0 - t = r^2$, hence $\inf_{\mathbf{B}(0, r)} \Phi(t, \cdot) \geq cr^{-\mathbf{n}}$. So by the above inequality

$$(B.1) \quad r^{-\mathbf{n}} \mu_t(\mathbf{B}(0, r)) \leq C \int_{\mathbf{B}(0, \varrho(t))} \Phi(0, x) \, d\mu_0(x).$$

By definition of Φ and $s_0 = t + r^2 \geq t = (8\mathbf{n})^{-1}\varrho(t)^2$ we have $\Phi(0, x) \leq C(\varrho(t))^{-\mathbf{n}}$ for all $x \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$. Thus (B.1) establishes the result. \square

Next we show that under the assumption of certain measure bounds if (1.4) is satisfied at one point a slightly weaker estimate is satisfied in a small neighbourhood.

B.2 Lemma. *There exists a $C \in (1, \infty)$ such that the following holds:*

Let $\varrho, M, \kappa \in (0, \infty)$, $\Lambda \in [1, \infty)$, $\delta \in (0, 1/(C\Lambda))$, $s_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and let μ be a Radon measure on $\mathbb{R}^{\mathbf{n}+\mathbf{k}}$. Suppose we have

$$(B.2) \quad \mu(\mathbf{B}(y_0, C\Lambda\varrho)) \leq M\varrho^{\mathbf{n}},$$

$$(B.3) \quad \int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(s_0, y_0)} \varphi_{(s_0, y_0), \Lambda\varrho}(s_0 - \varrho^2, x) \, d\mu(x) \leq 1 + \kappa.$$

Then for all $(s, y) \in (s_0 - \delta^2\varrho^2, s_0] \times \mathbf{B}(y_0, \delta\varrho)$ we have

$$\int_{\mathbb{R}^{\mathbf{n}+\mathbf{k}}} \Phi_{(s, y)} \varphi_{(s, y), \Lambda\varrho}(s_0 - \varrho^2, x) \, d\mu(x) \leq 1 + \kappa + CM\Lambda\delta.$$

Proof. We may assume $s_0 = 0$, $y_0 = 0$ and $\varrho = 1$. Fix $(s, y) \in (-\delta^2, 0] \times \mathbf{B}(0, \delta)$. Note that $\text{spt}(\varphi_{(s, y), \Lambda}(-1, \cdot)) \subset \mathbf{B}(0, (2\mathbf{n}+1)\Lambda)$. Let $x \in \mathbf{B}(0, (2\mathbf{n}+1)\Lambda)$. Direct calculations yield

$$1 \leq (s+1)^{-\mathbf{n}/2} \leq (s+1)^{-\mathbf{n}} \leq 1 - Cs \leq 1 + C\delta$$

$$\exp\left(\left|\frac{|x|^2}{4} - \frac{|x-y|^2}{4(s+1)}\right|\right) \leq \exp(C(\Lambda|y| + \Lambda^2|s|)) \leq 1 + C\Lambda\delta,$$

where we used $\delta \leq (C\Lambda)^{-1}$. Thus we have

$$|\Phi_{(0, 0)}(-1, x) - \Phi_{(s, y)}(-1, x)| \leq C\Lambda\delta$$

$$|\varphi_{(0, 0), \Lambda}(-1, x) - \varphi_{(s, y), \Lambda}(-1, x)| \leq C\delta$$

Combined with (B.2) and (B.3) this yields the result. \square

B.3 Remark. Here we want to derive Theorem 2.11 from Ilmanen’s work [Ilm94]. In case $U_i \equiv U$ the result directly follows from the proof of [Ilm94, Thm. 7.1]. Now consider the general case. We can find a subsequence $\lambda_1 : \mathbb{N} \rightarrow \mathbb{N}$ and a Brakke flow $(\nu_t^1)_{t \in [t_1, t_2]}$ in U_1 such that $\lim_{j \rightarrow \infty} \mu_t^{\lambda_1(j)}(\phi) = \nu_t^1(\phi)$ for all $\phi \in \mathcal{C}_c^0(U_1)$, for all $t \in [t_1, t_2]$. Inductively for all $l \in \mathbb{N}$, $l \geq 2$ we can find a subsequence $\lambda_l : \mathbb{N} \rightarrow \lambda_{l-1}[\mathbb{N}]$ and a Brakke flow $(\nu_t^l)_{t \in [t_1, t_2]}$ in U_l such that $\lambda_l(1) \geq l$ and

$$\lim_{j \rightarrow \infty} \mu_t^{\lambda_l(j)}(\phi) = \nu_t^l(\phi) \quad \text{for all } \phi \in \mathcal{C}_c^0(U_l)$$

for all $t \in [t_1, t_2]$. In particular we have $\nu_t^{l_2} \llcorner U_{l_0} = \nu_t^{l_1} \llcorner U_{l_0}$ for all $l_0 \leq l_1 \leq l_2$ and all $t \in [t_1, t_2]$. Then $\mu_t(\phi) := \lim_{l \rightarrow \infty} \nu_t^l(\phi|_{U_l})$ is well defined for all $\phi \in \mathcal{C}_c^0(U)$ and gives the desired Brakke flow on U . With $\sigma(j) = \lambda_j(j)$ this establishes the result.

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